DESIGN OF LIMITED STATE FEEDBACK CONTROLLERS FOR LINEAR MULTIVARIABLE SYSTEMS

A thesis
presented for the degree of
Doctor of Philosophy in Electrical Engineering
in the
University of Canterbury,
Christchurch, New Zealand.

by
SAN SHING CHOI B.E. (HONS.)

University of Canterbury
June 1976

ABSTRACT

The control of linear multivariable systems (LMS) where only some of the state variables are directly measurable is considered. The control configurations adopted employ feedback from the measurable state variables, i.e., the system outputs, via multivariable dynamic compensators. The design problem of determining the compensator parameters is approached via the following methods:

(1) The minimization of quadratic performance indices in the state and control variables, i.e., the optimal control method.

(2) The positioning of the closed-loop system poles, i.e., the pole placement (or modal control) method.

(3) The realization of appropriate state feedback laws through the use of observers.

The optimal output feedback control of essentially noise-free LMS is first examined. A gradient-type solution algorithm is developed that appears to be more efficient computationally than previous techniques; a modified algorithm for handling open-loop unstable LMS is also described. The treatment is then generalized to include cases where the LMS contains appreciable amounts of process and measurement noise; both stationary and non-stationary stochastic problems are considered.

Pole placement via output feedback is next examined as a possible alternative to the optimal control approach.

To this end, unrestricted-rank pole placement techniques are developed which enable the closed-loop poles to be positioned either arbitrarily close to specified locations, or within prescribed regions of the complex plane. Unlike previous work, the new techniques enable exact pole placement to be achieved with dynamic compensators having the lowest possible order. Consideration is then given to the more general problems of achieving exact (or approximate) pole placement while minimizing

(a) quadratic performance indices in the state and control variables,

(b) pole sensitivities to small or large system parameter variations, and

(c) steady-state following errors due to measurable and unmeasurable disturbances.

Finally, the construction of minimal-order observers is formulated as a static optimization problem for which a gradient-type solution technique is proposed. The suitability of using such observers to realize state feedback laws for achieving optimal control, pole placement or decoupling is also examined.

TABLE OF CONTENTS

ACKNOWLEDGEMENTS

I am especially grateful to my supervisor, Dr H.R. Sirisena, for his guidance and constant encouragement. His assistance over the duration of this project has been invaluable.

I am also very grateful to my parents for their most generous financial assistance which made possible my stay in Canterbury, and to my cousin, Miss K.E. Pang, for her patience and encouragement.

In the course of this work, considerable use was made of the EAI 640 digital computer at the Electrical Engineering Department and the access to these facilities is gratefully acknowledged.

GLOSSARY

Symbols, Abbreviations and Definitions

Unless otherwise specified, symbols have the meanings given below.

$\|X\|$	:	norm of X
$\|x\|$	:	modulus of x
X'	:	transpose of X
$\oplus$	:	direct sum
j	:	square root of -1, $j^2 = -1$
I_n	:	the $n \times n$ identity matrix
$0_{m,n}$	:	the $m \times n$ null matrix
$X > 0$	:	matrix X is positive definite
$X < 0$	:	matrix X is negative definite
$X \geqslant 0$	:	matrix X is non-negative definite
$\triangleq$	:	is defined as
$E\{x\}$	:	expected value of x
§	:	section
wrt	:	with respect to
$\delta(t)$	:	the delta function
$\lambda_i(X)$	:	the i^{th} eigenvalue of matrix X
$\rho[X] = \text{rank}\ [X]$	:	rank of X
$tr\{X\}$	:	the trace of X
$Re\{x\}$	:	the real part of x
$Im\{x\}$	:	the imaginary part of x
$x(t) \varepsilon R^n$	:	x(t) belongs to the n^{th} dimensional real space

$\Longleftrightarrow$	:	if and only if
$\Longrightarrow$	:	implies
$\Longleftarrow$	:	is implied by
RHP	:	Right half of complex plane
LHP	:	Left half of complex plane
RHS	:	Right hand side
LHS	:	Left hand side
LMS	:	Linear multivariable system
MIMO	:	Multi-input multi-output
MISO	:	Multi-input single-output
SIMO	:	Single-input multi-output
det	:	determinant
adj(X)	:	adjoint of X
dim(x)	:	dimension of x
$col[x_1,x_2,...x_m]$	:	columns of $x_1,x_2,...,x_m$
$max(x_1,x_2,...,x_m)$	:	the maximum value contained in the set $x_1,x_2,...,x_m$
$min(x_1,x_2,...,x_m)$	:	the minimum value contained in the set $x_1,x_2,...,x_m$

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

1.1 EARLY DEVELOPMENTS IN CONTROL ENGINEERING

Feedback control systems have been in existence for thousands of years. However, until comparatively recently, such systems were not designed formally but were rather the product of mechanical ingenuity. One of the earliest attempts at analysis was by Maxwell [1] who studied the phenomenon of hunting (instability) in flyball governors. About the same period, Routh discovered his famous stability criterion and by the end of the 19th century, several basic concepts in control engineering, e.g. control loop, feedback, controller with dynamic elements, etc. had begun to emerge [15].

1.1.1 Classical trial-and-error techniques

The demand for more accurate servomechanisms in connection with fire-control and other applications in World War II resulted in considerable research efforts being directed towards control system design. Tremendous strides were made during this period in extending analytical and design techniques as well as in the actual construction of high performance control systems. Thus, Harris [2] in 1941 and Hall [3] in 1943 adapted the work of Nyquist [4] on the feedback amplifier to the analysis and design of servomechanisms in the frequency (ω-) domain, while Bode [5], MacColl [6] and others similarly adapted the results of Bode on minimum phase electrical networks. A second approach to control system analysis and design is via Laplace transforms. This so called s-domain approach was pioneered by Gardner and Barnes [7] and culminated in the root-locus-method (RLM) introduced by Evans [8] about 1948.

The conventional ω-domain and s-domain techniques have one thing in common in that a trial-and-error design procedure is involved. The design process begins with an educated guess as to the form of a suitable controller whose parameters are tentatively chosen on the basis of a partial analysis. If the complete analysis that follows shows that the performance of the control system does not meet specifications, then the design is modified in a manner governed largely by the designer's experience and intuition. This is again followed by a performance analysis. The process is repeated until the specifications are satisfied. The design tools available to the designer are either graphical or analytical, e.g. Root Loci, Nyquist and Bode plots.

1.1.2 Analytical design techniques

In contrast to the trial-and-error approach, alternative design procedures, called analytical design techniques, have been explored since the early 1940's. Early work in this area included that by Weiner, Hall [3], Bode and Shannon [9], James, Nichols and Phillips [10], Newton, Gould and Kaiser [11]. The techniques are one step beyond the trial-and-error stage because the methods proceed directly from the problem specifications to the design without the need for human intuition. The design procedure begins with a suitably specified performance index which gives a qualitative measure for the performance of the system. Unlike the trial-and-error approach, no explicit statement concerning the degree of stability is required except that all solutions must result in the system being stable and the controller being realizable. The effects of noise on system performance were also considered. An excellent text which describes these techniques is [11].

1.1.3 Direct synthesis method of Guillemin and Truxal

Another direct synthesis method is that due to Guillemin and Truxal [12]. The design procedure begins with a reduction of the design specifications to a desired closed-loop transfer function characterized by its pole-zero configuration. A compensator is then designed where the unwanted plant poles and zeros are simply cancelled out. However, specifying the desirable closed-loop transfer function often over-defines the design problem, placing unnecessary and undesirable restrictions upon the designer. Furthermore, pole-zero cancellation without regard to internal state variables could result in the system being uncontrollable and unobservable [13].

The trial-and-error, the analytical and the direct synthesis design techniques described above have been developed mainly towards the control of single-input, single-output (SISO) systems. By the end of World War II, this so called *classical control theory* was quite well developed. It was therefore natural to expect that the next stage of the control theory evolution would be in the area of multi-input, multi-output (MIMO) control system design. Indeed, attempts to solve the MIMO control problem had been reported as early as 1938 in a paper by Voznesenskii [42].

1.2 LINEAR MULTIVARIABLE CONTROL SYSTEM DESIGN

A fundamental characteristic of multivariable systems is the coupling or interaction between the input and output variables, in that one input affects more than one output. A consequence of such interaction often is a reduction in the stability margins of system operation [16]. Hence, the possibility of designing a succession of feedback loops one at a time using well-established classical feedback theory for a general MIMO system is unlikely to be successful.

To overcome this problem, it has been proposed that the MIMO system be first decoupled into a number of SISO subsystems by appropriate compensation. This is discussed in the next section.

1.2.1 Non-interactive design technique

Consider the general multivariable system shown in Fig. 1.1 with an equal number of inputs in $u(s)$ and outputs in $y(s)$.

The design objective may be simply stated as: find a suitable $K(s)$ such that $G(s)K(s)$ is a diagonal matrix.

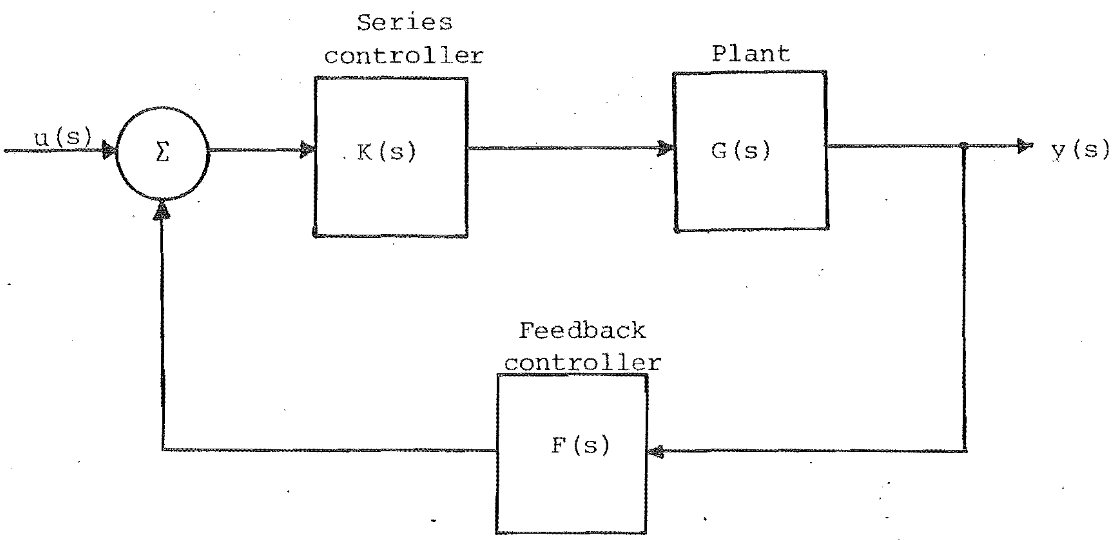

Fig. 1.1 A general multivariable control system

When this occurs, *non-interaction* is said to have been achieved. Clearly, this results in the MIMO system being decoupled into a system of several SISO systems. Then, the classical design techniques mentioned in §1.1 may be used to determine an appropriate diagonal matrix F(s) so that the resulting closed-loop system has the desired performance characteristics. This completes the design procedure.

It was claimed [42] that the non-interacting problem posed above was first solved by the Russian researcher Voznesenskii in 1938. However, the first reported work in English was that by Boksenbom and Hood [17] in 1949 where a non-interacting jet engine was synthesized. Subsequent applications of this technique to other practical control problems can be found elsewhere, e.g. [18], [19].

Boksenbom and Hood have approached the problem via a frequency domain technique. Recently, the problem has been considered using state-space (time-domain) formulation (see e.g. [20]) which is in a form more convenient for programming on a digital computer. Although some contributions in this area have been made by the author and his project supervisor (see the paper by Sirisena and Choi [21]), the details of this work have not been included in this thesis because it is felt that non-interactive design technique is likely to be of limited applicability for reasons listed in Table I.

In view of the disadvantages and difficulties met in designing perfectly non-interacting LMS, the inverse Nyquist array (INA) method has been devised by Rosenbrock

Advantage	Disadvantages
(A1) Conceptually simple.	(D1) Complicated controller is usually required to achieve exact non-interaction [16], [22]. (D2) Excessive degree of design freedom is used to make $G(s)K(s)$ diagonal leaving little design freedom for improving system performance [16]. (D3) If $\det[G(s)]$ has RHP zeros, then the resulting design gives unstable or poor control [22].

Table I

[24] where only "approximate" non-interaction is aimed for. The technique is described in the next section.

1.2.2 Inverse Nyquist array method

Consider again the LMS depicted in Fig. 1.1. Denote $\hat{Q}(s) = [G(s)K(s)]^{-1}$ and the i,j element of $\hat{Q}(s)$ by $\hat{q}_{ij}(s)$. Assume that $\hat{Q}(s)$ is of dimension $m \times m$.

In the INA method, $K(s)$ is chosen using an interactive and iterative process such that $\hat{Q}(s)$ is diagonal dominant, i.e.

$$|\hat{q}_{ii}(s)| >> \sum_{\substack{i=1\\ \neq j}}^{m} |\hat{q}_{ij}(s)| \triangleq d_i(s) \quad , \qquad i = 1,\dots,m$$

Diagonal dominance may be attained if the following steps are followed [24]: initialize design procedure by guessing the form and values of $K(s)$, then

Step (i): Plot $\hat{q}_{ii}(s)$, $i = 1,\dots,m$ for a range of values of $s = j\omega$.

Step (ii): On the locus $\hat{q}_{ii}(s)$, construct a set of circles each of radius $d_i(s)$ centered at the appropriate s. This is repeated for $i = 1,\ldots,m$. A typical plot is shown in Fig. 1.2.

Step (iii): If any $d_i(s)$ is compatible (or large) compared to $|\hat{q}_{ii}(s)|$, then diagonal dominance has not been achieved for the i^{th} input/output pair, in which case, K(s) is usually modified in a manner very much governed by the designer's intuition and experience. The procedure is then repeated.

When diagonal dominance has been achieved, then suppose the band of circles centred on $\hat{q}_{ii}(s)$ (the "Gershgorin band") does not touch the segment of the negative real axis between the origin and the point $-k_i$ (see Fig. 1.2), all i. The design of feedback control system may then be completed on the basis of a set of individual single loops, see Fig. 1.3, with gains f_i, $i = 1,\ldots,m$. Furthermore, it has been shown [24] that so long as the feedback gain f_i are chosen within the range $-k_i \leqslant f_i \leqslant 0$, then the overall control system is stable. Therefore, the control system has high integrity [24] in that should there be a hardware failure in the i^{th} feedback loop, making $f_i = 0$, the closed-loop system is still stable.

Most of the work on INA has been carried out at the University of Manchester Institute of Science and Technology (UMIST). Applications of this method to practical control problems have also been reported, see e.g. [25] - [27]. As of this writing, the potential of this method is unclear

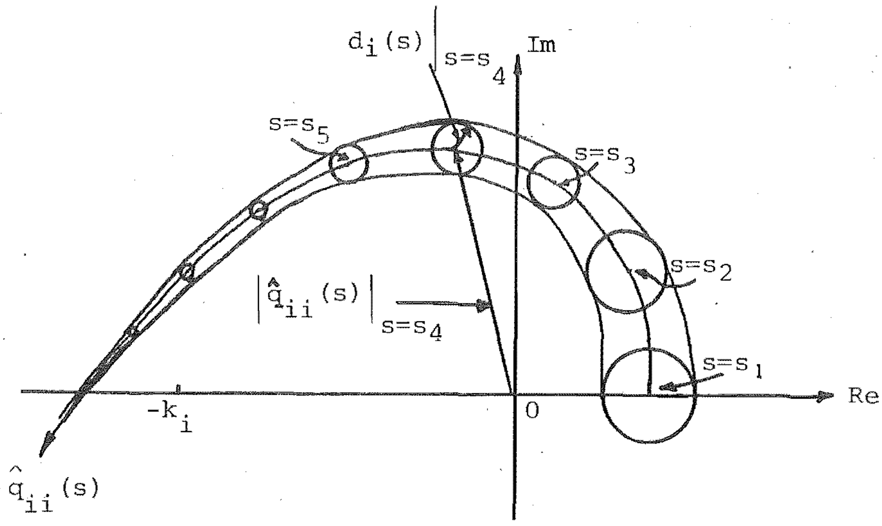

Fig. 1.2 A typical inverse Nyquist array plot

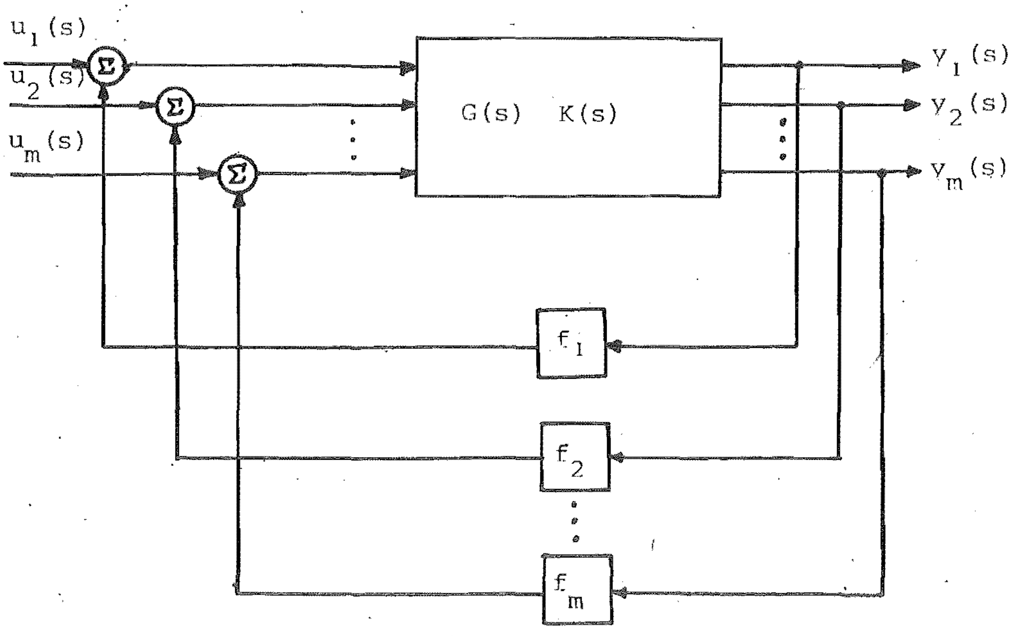

Fig. 1.3 The complete control system

because apart from several theoretical problems that still remain to be solved [28], [29], the method suffers from the shortcomings summarized in Table II.

Advantages		Disadvantages	
(A1)	Only a limited amount of experimental data on plant is needed.	(D1)	It is not truly a synthesis method.
(A2)	It is insensitive to plant model inaccuracy.	(D2)	The method requires design experience if it is to be used successfully.
(A3)	It is a high integrity design technique with respect to the opening of feedback loops.	(D3)	The method is interactive and iterative.
(A4)	It can handle "standard" forms of engineering performance specifications.		

Table II

1.2.3 Optimal control approach

It is perhaps in recognition of the disadvantages listed in Table I that prompted researchers to abandon the aim of achieving perfect non-interaction. It was also felt [23] that in view of the reason (D2) given in the same table, the performance of coupled LMS may be made superior to that of decoupled LMS. Towards this end, one of the most powerful design techniques that has been developed to date deals with the design of the optimal feedback system for a linear, possibly time-varying plant with quadratic performance index [30]. The pioneering work in this area was done by Kalman [31]. Several texts have since been written on this particular control topic, see eg. [32] - [34]. Suppose *all the state variables are directly measurable*, then the linear optimal control problem is formulated as follows.

Given the LMS

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{1-1}$$

$x(t) \varepsilon R^n$, $u(t) \varepsilon R^m$ are the state and control vectors respectively. $A(\cdot)$ and $B(\cdot)$ are matrices of appropriate dimensions.

The design objective is to minimize a performance index $J(u)$ where

$$J(u) = \tfrac{1}{2}x'(T)Sx(T) + \tfrac{1}{2}\int_0^T x'(t)Q(t)x(t) + u'(t)R(t)u(t)dt \tag{1-2}$$

with the assumption that $Q(t) \geqslant 0$, $R(t) > 0$ and $S \geqslant 0$. T is the terminal time. (1-2) is a generalization of the classical integral-square error performance criterion [11].

The solution of the problem is well known and can be derived via Hamilton-Jacobi-Bellman theory. It can be shown that the optimal control $u^*(t)$ which minimizes (1-2) exists, is unique and is given by the equation

$$u^*(t) = -R^{-1}(t)B'(t)K(t)x^*(t) \tag{1-3}$$

where $K(t) = K'(t) \geqslant 0$ is the solution of the matrix Riccati equation

$$\dot{K}(t) = -K(t)A(t) - A'(t)K(t) + K(t)B(t)R^{-1}(t)B'(t)K(t) - Q(t) \tag{1-4}$$

with boundary condition

$$K(T) = S \tag{1-5}$$

The derivation of these results is omitted but can be found in [31] - [34].

If (1-1) is time-invariant and Q,R are constant matrices, $S = 0_{n,n}$ and $T \to \infty$, the optimal feedback law is linear and time-invariant [31], i.e.

$$u^*(t) = - R^{-1}B'\hat{K}x^*(t) \tag{1-6}$$

where $\hat{K}' = \hat{K} > 0$ is the solution of the algebraic Riccati equation

$$0 = - \hat{K}A - A'\hat{K} + \hat{K}BR^{-1}B'\hat{K} - Q \tag{1-7}$$

The properties of the time-invariant optimal closed-loop system described above have also been investigated, see e.g. [32]. By considering the Nyquist plot of the optimal closed-loop system, it has been established that the gain-margin of the closed-loop system is theoretically infinite, while the phase margin is at least 60° [32]. Of course, no physical system can have infinite gain margin due to such parasitic effects as stray capacitance, time delay, etc. However, even if the linear models are only approximate representations of the real systems, the gain margin of the optimal closed-loop system may still be very good. The effects of non-linearities and time-delays in the optimal closed-loop systems have also been studied [32].

The design techniques using the state feedback laws (1-3) or (1-6) have since been applied to practical design problems. For a survey on this, see e.g. [30].

It must be remembered that in the formulation of the linear optimal control problem, it is assumed that the plant is described exactly by (1-1) and that all the state variables are available for use in the control law (1-3) or

(1-6). There are many practical situations where these assumptions are not valid [14]. The advantages and disadvantages of the optimal control technique are summarized in Table III.

Advantages	Disadvantages
(A1) A true synthesis method.	(D1) Requires accurate plant model.
(A2) Solution of the problem is explicitly known and easily computed.	(D2) The complete state vector must be available for the implementation of the optimal control law.
(A3) Economic type performance specification.	(D3) Under certain conditions, the optimal controller may be of low integrity [14].

Table III

Much research effort has been directed towards overcoming these shortcomings. Thus, model inaccuracy motivates the search for more efficient techniques in system identification [35] and the design of robust controllers, see e.g. [36]. Also, Luenberger observer theory [37] has been developed to handle cases where not all state variables of a noise-free plant can be measured directly: the Luenberger observer estimates the missing states asymptotically as $t \to \infty$. For plants operating in a noisy environment, the Luenberger observer is replaced by a Kalman-Bucy filter [38]. Although these results are pleasing and do provide a way round the problem, the resulting control scheme tends to be unnecessarily complex due to the high order observer (or filter) present in the feedback loop.

The problem of inaccessible states has been attacked more directly by Levine and Athans [40] who adopt a fixed-configuration approach with feedback from the measurable states in what amounts to a generalization of classical analytical design techniques [11]. Subsequent extensions of this approach (see Chapters 2 and 3 of this thesis) provide a means for avoiding the use of high-order observers or filters.

1.2.4 Pole placement approach

Another synthesis method for time-invariant LMS that has captured much recent research interest is the so called pole placement (or modal control) approach.

Consider now the time-invariant linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_o \tag{1-8}$$

$x(t) \varepsilon R^n$, $u(t) \varepsilon R^m$ and the general state feedback law

$$u(t) = Fx(t) . \tag{1-9}$$

Therefore the closed-loop system is given by

$$\dot{x}(t) = (A + BF)x(t) \tag{1-10}$$

and the solution of the differential equation is

$$X(s) = \frac{\mathrm{adj}(sI-A-BF)}{|sI-A-BF|} X_o(s) \tag{1-11}$$

then the pole placement problem may be stated simply as: find an appropriate F such that the n roots (λ) of the characteristic polynomial $|\lambda I-A-BF| = 0$, i.e. the poles of the closed-loop system (1-10), assumed some arbitrary preassigned values.

In the design of SISO control system, for example, a gain (f) in the closed-loop system is adjusted so that the poles are shifted to desirable positions of the complex plane. A common method used is the RLM which is in fact a graphical means of plotting the pole positions as a function of f. However, the process can become extremely tedious when it involves more than one parameter. Therefore, there is a need to develop alternative techniques for handling pole-placement problems. Indeed, there has been much research activity in this area during the last few years.

The solution of the pole placement problem posed above is guaranteed to exist provided the plant (1-8) is controllable [39]. For situations where not all the state variables are available, an observer may be constructed to estimate the inaccessible states. However, the same criticism made earlier in §1.2.3 concerning the observer order also applies in this case. A more practical approach to the pole placement problem is to use feedback from those state variables which are directly measurable. Several design techniques to do this are currently available in the literature, see e.g. [41]. However, unlike the technique developed in Chapter 4, they do not, in general, result in the minimal-order controller being obtained.

The pole placement problem is also seen to be, in part, a generalization of the direct synthesis method of Guillemin and Truxal (GT) described in §1.1.3 to MIMO systems. Unlike the GT technique, however, the pole placement problem places no restriction on the numerator of the transfer matrix. The extra degrees of design freedom obtained as a result may be used to attain other design objectives. This will be described in Chapters 4, 5 and 6.

1.3 THESIS ORGANISATION

This thesis is devoted to the design of linear multivariable control systems using the modern state-space approach. Topics investigated include the optimal control of deterministic and stochastic systems, the pole placement problem, the servomechanism problem and the design of reduced-order observers. Research efforts are directed, in particular, towards developing efficient computational techniques which can be conveniently programmed on a digital computer. All expressions used in the computational procedures are also derived.

Perhaps most importantly, it will be seen that *all the design techniques described in this thesis employ feedback from only the directly measurable state variables, i.e. the plant outputs. Therefore, the practical difficulty in measuring or estimating the missing state variables for control purposes never arises.* Also, the controllers considered here are usually of lower order than that of Luenberger observers (for the deterministic case) or Kalman-Bucy filters (for the stochastic case). Then, very much in the spirit of optimal control, the available design freedoms are made use of to obtain the "best" system performance. The degree of "goodness" is, of course, determined on the basis of the design specifications.

Chapter 2 begins with an investigation into the optimal control of essentially noise-free time-invariant systems using fixed-configuration compensators. The main contribution in this chapter is the development of a gradient-type algorithm for obtaining the optimal compensator parameters. Unlike all other available techniques, the proposed method avoids the solution of non-linear matrix equations while appearing to exhibit rapid convergence. Consideration is also given to the optimal control of open-loop unstable plants for which a modified gradient-type algorithm is proposed.

The results are then extended, in Chapter 3, to the optimal control of linear systems having significant measurement and process noise. For the stationary and non-stationary stochastic problems considered in this chapter, it will be shown that the cost and its gradients with respect to the compensator parameters are not unlike those obtained for the deterministic problem considered in Chapter 2. Therefore, a gradient-type algorithm similar to that described in Chapter 2 may be used to obtain the optimal compensator.

From §1.2.4, it is seen that another possible synthesis method is via the technique of pole placement. Thus, in Chapter 4, a new (unrestricted-rank) pole placement technique is presented. It is shown how the pole placement problem can be formulated as a static optimization problem. The technique is computationally superior when compared to existing methods and always enables exact pole placement to be achieved with a minimal-order compensator. Consideration is also given to the more general problem of achieving exact or approximate pole placement while minimizing a quadratic performance index in the state and control variables. This is achieved by combining the results obtained for exact pole placement with that contained in §2.2.

It may be argued that in many practical situations, the exact positioning of the closed-loop poles is perhaps only of secondary importance; it may suffice to position them within a prescribed region of the complex plane. Moreover, this less stringent design requirement on closed-loop poles may be satisfied with compensators of lower order than that required for exact pole placement. Motivated by these reasons, an algorithm to achieve this design aim is developed in §4.4.

In Chapter 5, the sensitivity of closed-loop poles to plant parameter variations is investigated. Two cases are considered. The first case is concerned with situations where the parameter variations are small compared to the nominal values of the plant parameters. A suitable measure of the pole sensitivity is defined and it is shown how exact pole placement can be achieved at the nominal values of the parameters while minimizing the sensitivity measure. The second case deals with the situations where the parameter variations are not necessarily small compared to their nominal values. An alternative design algorithm based on a new measure of pole sensitivity is then proposed.

In Chapter 6, the pole placement technique of Chapter 4 is extended to the design of dynamic compensators for servomechanisms having measurable and unmeasurable constant inputs. The performance of the closed-loop system is assessed not only in terms of its transient response characteristics but also the steady-state error that may exist between the output and the reference inputs. Control on the system transient response characteristics is through suitable placement of the closed-loop poles. A two-step design procedure is then proposed. In the first step, only the unmeasurable constant inputs are considered. A dynamic compensator is constructed via a function minimization procedure such that optimal transient and steady-state performances are obtained. It is then shown, in the second step of the design procedure, how the steady-state error due to the measurable constant inputs can be made exactly zero through the use of a feedforward controller. The conditions under which such a dynamic compensator-feedforward controller exists are included in this chapter.

Consideration is also given to the case where the inputs are actually time-varying Markov processes. The performance of the servomechanism is assessed by a quadratic criterion. It is then shown that the design problem is actually mathematically equivalent to the stationary stochastic regulator problem considered in Chapter 3. Therefore, the gradient-type solution algorithm can also be used to obtain the optimal compensator parameters.

Chapter 7 deals with the construction of minimal-order observers. By means of geometric arguments, the observer design problem is reduced to a static optimization problem in certain observer parameters. A systematic procedure for designing minimal-order stable observers is proposed that is based on a new lower bound on the required observer order, a special canonical form for the observer matrix that ensures any prescribed degree of stability and a gradient-type function minimization algorithm. A modified procedure for designing minimal-order observers having arbitrarily specified poles is also described. Finally, the role of observers in implementing state feedback laws for pole placement, decoupling or minimizing quadratic performance indices is also considered.

In Chapter 8, contributions made in this thesis are summarized. Several areas for future research are also included.

1.4 PUBLICATIONS

Some of the works described in this thesis also appear in the following publications:

CHOI, S.S. and SIRISENA, H.R., "Computation of optimal output feedback gains for linear multivariable systems," IEEE Trans. Automat. Contr., vol. AC-19, pp.257-258, June 1974.

SIRISENA, H.R. and CHOI, S.S., "Design of optimal constrained dynamic compensators for linear stationary stochastic servomechanisms," Int. J. Contr., vol. 20, pp. 363-368, Sept. 1974.

SIRISENA, H.R. and CHOI, S.S., "Optimal pole placement in linear multivariable systems using dynamic output feedback," Int. J. Contr., vol. 21, pp.661-671, April 1975.

SIRISENA, H.R. and CHOI, S.S., "Pole placement in output feedback control systems for minimum sensitivity to plant parameter variations," Int. J. Contr., vol. 22, pp.129-140, July 1975.

SIRISENA, H.R. and CHOI, S.S., "Pole placement in prescribed regions of the complex plane using output feedback," IEEE Trans. Automat. Contr., vol. AC-20, pp.810-812, Dec. 1975.

SIRISENA, H.R. and CHOI, S.S., "Design of optimal constrained dynamic compensators for nonstationary linear stochastic systems," Int. J. Contr., to appear.

SIRISENA, H.R. and CHOI, S.S., "An algorithm for constructing minimal-order observers for linear functions of the state," Int. J. System Science, accepted for publication.

CHOI, S.S. and SIRISENA, H.R., "Computation of optimal output feedback controls for unstable linear multivariable systems," IEEE Trans. Automat. Contr., accepted for publication.

REFERENCES

[1] Maxwell, J.C., "On governors," Proc., Roy. Soc. (London), vol. 16, pp.270-283, 1868.

[2] Harris, H., "The analysis and design of servo-mechanism," OSRD Report, 454, Dec. 1941.

[3] Hall, A.C., Analysis and Synthesis of Linear Servomechanism, Technology Press, Cambridge, 1943.

[4] Nyquist, H., "Regeneration theory," Bell System Tech. J., vol. 11, pp.126-147, 1932.

[5] Bode, H.W., "Feedback amplifier design," Bell System Tech. J., vol. 19, pp.421-454, 1940.

[6] MacColl, L.A., "Fundamental Theory of Servomechanism," D. Van Nostrand Company, N.Y., 1945.

[7] Gardner, M.F. and Barnes, J.L., Transients in Linear Systems, John Wiley & Sons, N.Y., vol. 1, 1942.

[8] Evans, W.R.,"Graphical analysis of control systems," Trans. AIEE, Appl. Ind., vol. 67, pp.547-551, 1948.

[9] Bode, H.W. and Shannon, C.E., "A simplified derivation of linear least square and prediction theory," Proc. I.R.E., vol. 38, pp.417-424, 1950.

[10] James, H.M., Nichols, N.B. and Phillips, R.S., "Theory of Servomechanisms," McGraw-Hill Book Company, N.Y., 1947.

[11] Newton, G.C., Gould, L.A. and Kaiser, J.F., Analytical Design of Linear Feedback Controls, John Wiley & Sons, Inc., N.Y., 1957.

[12] Truxal, J.G., Control System Synthesis, McGraw-Hill, New York, 1955.

[13] Gilbert, E.G., "Controllability and observability in multivariable control systems," J. SIAM Control Ser. A, vol. 1(2), pp.128-151, 1963.

[14] Rosenbrock, H.H. and McMorran, P.D., "Good, bad or optimal?" IEEE Trans. Automat. Contr., vol. AC-16, pp.552-554, Dec. 1971.

[15] Fuller, A.T., "The early development of control theory. I," University of Cambridge report, CUED/F-CAMS/TR119(1976).

[16] MacFarlane, A.G.J., "A survey of some recent results in linear multivariable feedback theory," Automatica, vol. 8, pp.455-492, 1972.

[17] Boksenbom, A.S. and Hood, R., "General algebraic method applied to control analysis of complex engine types," Report NCA-TR-980, NACA, Washington D.C., 1949.

[18] Chatterjee, H.K., "Multivariable process control," Proc. IFAC, Moscow, 1960, Butterworths, 1961.

[19] Ergin, E.I. and Ling, C., "Development of a non-interacting controller for boilers," Proc. IFAC, Moscow, 1960, Butterworths, 1961.

[20] Gilbert, E.G., "The decoupling of multivariable systems by state feedback," J. SIAM Control, Vol. 7, pp.50-63, 1969.

[21] Sirisena, H.R. and Choi, S.S., "Minimal-order compensators for decoupling and arbitrary pole placement in linear multivariable systems," Int. J. Contr., to appear.

[22] Rosenbrock, H.H., "On the design of linear multi-variable control systems," Proc. 3rd Congress of IFAC, London, 1966.

[23] Mesarovic, M.D., "Control of multivariable systems," Proc. IFAC, Moscow, 1960, Butterworths, 1961.

[24] Rosenbrock, H.H., "Design of multivariable control systems using the inverse Nyquist array," Proc. I.E.E., vol. 116, pp.1929-1936, Nov. 1969.

[25] McMorran, P., "Design of a gas turbine controller using the inverse Nyquist method," Proc. I.E.E., vol. 117, pp.2050-2056, 1970.

[26] Munro, N., "Design of controllers for open-loop unstable multivariable systems using inverse Nyquist array," Proc. I.E.E., vol. 119, pp.1377-1382, 1972.

[27] Hawkins, D.J., "Pseudodiagonalization and the inverse Nyquist array method," Proc. I.E.E., vol. 119, pp.337-342, March 1972.

[28] Rosenbrock, H.H., "Progress in the design of multivariable control systems," Measurement and Control, vol. 4, pp.9-11, 1971.

[29] Rosenbrock, H.H., Computer-aided Control System Design, Academic Press, London, 1974.

[30] Athans, M., "The status of optimal control theory and applications for deterministic systems," IEEE Trans. Automat. Contr., vol. AC-11, pp.580-596, July 1966.

[31] Kalman, R.E., "Contributions to the theory of optimal control," Bol. Soc. Mat. Mex., Second Series, vol. 5, pp.102-119, 1960.

[32] Anderson, B.D.O. and Moore, J.B., Linear Optimal Control, Prentice-Hall, Englewood Cliffs, N.J., 1971.

[33] Athans, M. and Falb, P.L., Optimal Control, McGraw Hill, New York, 1966.

[34] Kwakernaak, H. and Sivan, R., Linear Optimal Control Systems, Wiley-Interscience, N.Y., 1972.

[35] Special Issue on System Identification and Time-series Analysis, IEEE Trans. Automat. Contr., vol. AC-19, Dec. 1974.

[36] Davison, E.J. and Goldenberg, A., "Robust control of a general servomechanism problem: the servo compensator," Automatica, vol. 11, pp.461-471, 1975.

[37] Luenberger, D.G., "Observers for multivariable systems," IEEE Trans. Automat. Contr., vol. AC-12, pp.190-197, Apr. 1966.

[38] Kalman, R.E. and Bucy, R.S., "New results in linear filtering and prediction theory," Trans. ASME J. Basic Eng., vol. 83D, pp.95-108, 1961.

[39] Wonham, W.M., "On pole assignment in multi-input controllable linear systems," IEEE Trans. Automat. Control, vol. AC-12, pp.660-665, Dec. 1967.

[40] Levine, W.S. and Athans, M., "On the determination of the optimal constant output feedback gains for linear multivariable systems," IEEE Trans. Automat. Contr., vol. AC-15, pp.44-48, Feb. 1970.

[41] Brasch, F.M. and Pearson, J.B., "Pole placement using dynamic compensators," IEEE Trans. Automat. Contr., vol. AC-15, pp.34-43, Feb. 1970.

[42] Tu, XU-YEN, "Theory of an harmonically acting control system with a large number of controlled variables," Proc. IFAC, Moscow, 1960, Butterworths, 1961.

CHAPTER 2

DESIGN OF TIME-INVARIANT DETERMINISTIC OUTPUT FEEDBACK CONTROL SYSTEMS USING QUADRATIC PERFORMANCE INDICES

2.1 INTRODUCTION

As pointed out in Chapter 1, one approach to the design of deterministic LMS is via optimization with respect to a quadratic performance index. It is well known that for this index the unconstrained optimal control involves feedback from all the state variables and this is, in general, impracticable because

(i) the optimal state feedback law may be unnecessarily complicated. Simpler controllers could be constructed for which the overall system performance is still within design specifications. Such occurrences are quite common, for it is well known that an optimally designed system tends to have excessive gain and phase margins, see e.g. [1] and [2],

(ii) not all the states can be measured directly. One possibility is to estimate the missing states, using a Luenberger-type observer [29]. However, the resulting controller may still be of unnecessarily high order. Moreover (see Chapter 8), the performance of the closed-loop system may not be entirely satisfactory because the observer estimates the states only asymptotically.

A more direct approach is that adopted by Levine and Athans [3] and generalized by Johnson and Athans [4]. In this approach, the controller is constrained to be a fixed-configuration (low-order) dynamic system. The complete closed-loop system is then as shown in Fig. 2.1.

The design problem is to find the optimal compensator parameters contained in (P, N, G, H) to minimize the chosen quadratic performance index. The compensator structure shown in Fig. 2.1 is quite general and includes the classical lag- and lead-type compensators, proportional-plus-integral controllers, etc. Notice that this problem is of the same type as the analytical design problem treated by Newton, Gould and Kaiser [5] and others. However, the classical solution technique via Laplace transforms are oriented towards hand calculations and are only suitable for low order SISO systems having just a few adjustable parameters. The design of high-order, multi-parameter systems is only feasible via computer-oriented state-space methods such as those developed in [3], [4].

The algorithm given in [3], [4] for computing the optimal compensator parameters is of the so-called *indirect type:* the necessary conditions for optimality are derived as a set of equations which are then solved iteratively. Unfortunately, a non-linear algebraic matrix equation has to be solved at every iteration, and this is computationally expensive. A variant of this algorithm [6] avoids the solution of non-linear equations but at the risk of divergence.

In this chapter, an iterative algorithm of the

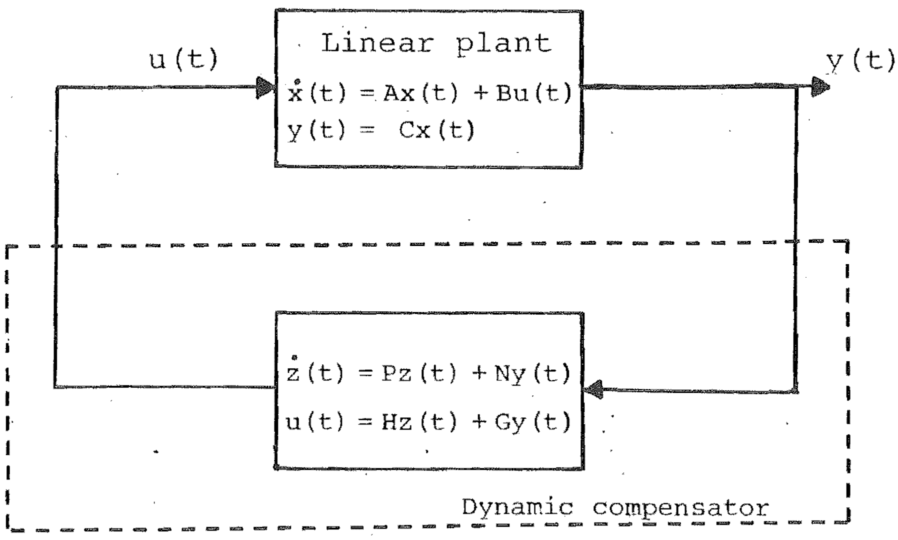

Fig. 2.1 The closed-loop linear multivariable system

direct type is presented which requires the solution of only linear equations while appearing to exhibit rapid convergence. The approach adopted is to directly minimize the performance index with respect to the compensator parameters using a gradient technique. Consideration is also given to the elimination of redundant compensator parameters by employing suitable canonical forms so as to minimize wasteful computation. The greatly improved computational efficiency of the new algorithm should enhance the practical usefulness of this design approach.

All the abovementioned algorithms require an initial guess of compensator parameters that stabilizes the closed-loop system. If the plant is open-loop stable, clearly there is no problem in doing so. However, the choice of such stabilizing compensator parameters becomes difficult if the plant is open-loop unstable. A modified form of the direct-type algorithm mentioned earlier that circumvents this difficulty is described in §2.3.

Although only the time-invariant, infinite time regulator problem is considered in this chapter, the results can be extended to the time-varying finite-time case as will be shown in Chapter 3.

2.2 OPTIMAL OUTPUT FEEDBACK PROBLEM

2.2.1 Problem formulation

A problem formulation similar to that in [4] is adopted. It is assumed that the controllable, observable plant under consideration is deterministic, linear and can be described by the differential equations

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_o \\ y(t) &= Cx(t) \end{aligned}\right\} \tag{2-1}$$

where $x(t)\varepsilon R^n$, $u(t)\varepsilon R^m$ and $y(t)\varepsilon R^r$ are the plant state, control and output vectors, respectively. The constant matrices A, B, C are appropriately dimensioned with $\rho[C] = r$.

The compensator, of given order p where $0 \leqslant p < n-r$, has the structure

$$\left.\begin{aligned} \dot{z}(t) &= Pz(t) + Ny(t), \quad z(0) = z_o \\ u(t) &= Hz(t) + Gy(t) \end{aligned}\right\} \tag{2-2}$$

The schematic of the complete closed-loop system is shown in Fig. 2.2.

The parameters to be determined are contained in the appropriately dimensioned matrices P, N, G and H and the vector z_o.

Equations (2-1) and (2-2) can be written more compactly as

$$\left.\begin{aligned} \dot{\overline{x}} &\triangleq (\overline{A} + \overline{B}F\overline{C})\overline{x}(t), \quad \overline{x}(0) \triangleq \overline{x}_o = \begin{bmatrix} x_o \\ z_o \end{bmatrix} \\ &\triangleq \overline{A}_o\overline{x}(t) \end{aligned}\right\} \tag{2-3}$$

where

$$\left.\begin{aligned} &\overline{x}(t) \triangleq \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}, \quad \overline{A} \triangleq \begin{bmatrix} A & 0_{n,p} \\ 0_{p,n} & 0_{p,p} \end{bmatrix}, \quad \overline{B} \triangleq \begin{bmatrix} B & 0_{n,p} \\ 0_{p,m} & I_p \end{bmatrix}, \\ &F \triangleq \begin{bmatrix} G & H \\ N & P \end{bmatrix}, \quad \overline{C} \triangleq \begin{bmatrix} C & 0_{r,p} \\ 0_{p,n} & I_p \end{bmatrix}. \end{aligned}\right\} \tag{2-4}$$

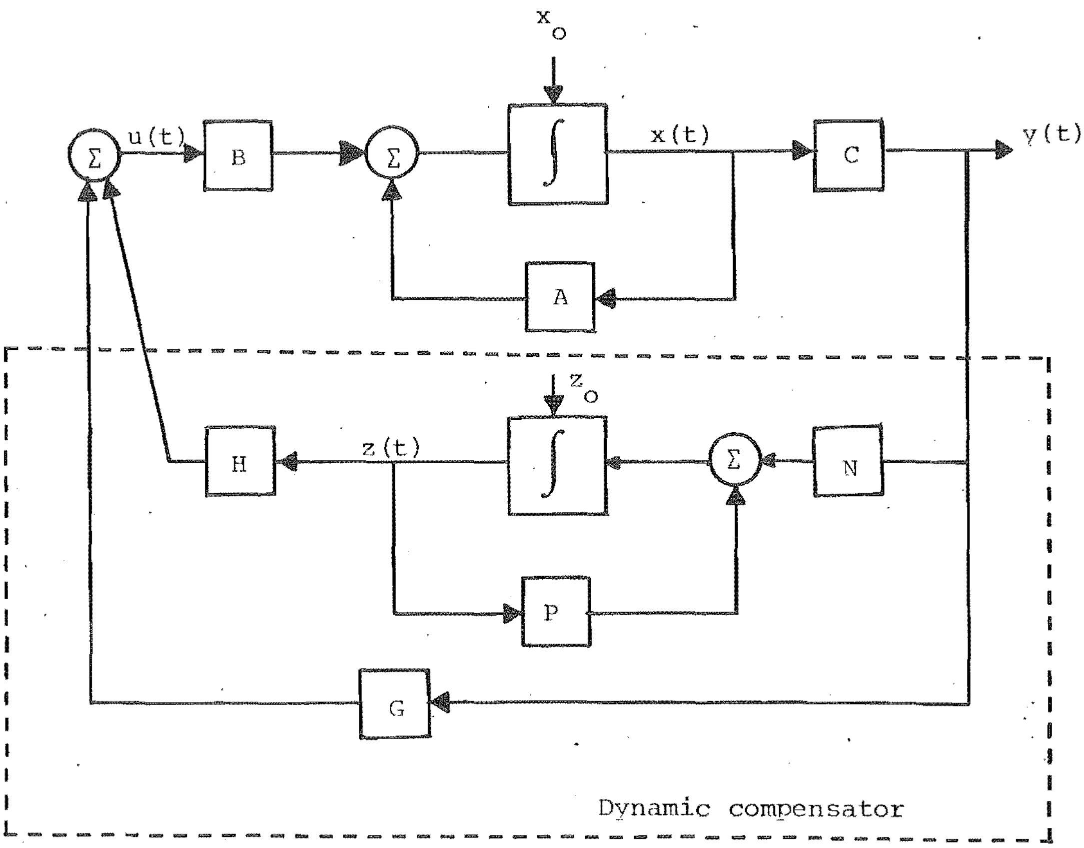

Fig. 2.2 The linear multivariable control system

The next step is to find a suitable performance index for the composite system (2-3). Suppose the following quadratic performance index is used,

$$\bar{J}(F) = \tfrac{1}{2}\int_0^\infty \bar{x}'(t)Q\bar{x}(t) + u'(t)R_1u(t) + \dot{z}'(t)R_2\dot{z}(t)dt \qquad (2\text{-}5)$$

where $Q \geqslant 0$, $R_1 > 0$ and $R_2 > 0$. Then from (2-3) and (2-4),

$$\bar{J}(F) = \tfrac{1}{2}\int_0^\infty \bar{x}'(t)[Q + \bar{C}'F'\bar{R}F\bar{C}]\bar{x}(t)dt \qquad (2\text{-}6)$$

and where

$$\bar{R} \triangleq \begin{bmatrix} R_1 & 0_{m,p} \\ 0_{p,m} & R_2 \end{bmatrix} \qquad (2\text{-}7)$$

Notice that in (2-5), quadratic weightings are placed not only on $\bar{x}(t)$ but also on the plant and compensator inputs; $u(t)$, $\dot{z}(t)$. This is in order to limit the compensator gains so as to satisfy physical constraints and also to limit noise transmission. The integrand in (2-5) differs slightly, though not materially, from that adopted in [4].

Now from (2-3),

$$\bar{x}(t) = \Phi(t,0)\bar{x}(0) \qquad (2\text{-}8)$$

where $\Phi(t,0)$ is the transition matrix given by

$$\Phi(t,0) = e^{\bar{A}_o t} \qquad (2\text{-}9)$$

Substitution of (2-8) into (2-6) gives

$$\bar{J}(F) = \tfrac{1}{2}\int_0^\infty \bar{x}'(0)\Phi'(t,0)[Q + \bar{C}'F'\bar{R}F\bar{C}]\Phi(t,0)\bar{x}(0)dt \qquad (2\text{-}10)$$

The optimal output feedback problem may thus be stated as follows:

Problem statement (S1)

Given the plant (2-1) and the compensator (2-2) of given order p, find the parameter matrix F such that the quadratic performance index (2-10) is minimized.

Remarks

(R1). It is assumed that the compensated system (2-3) is stable. Otherwise $\bar{J}(F)$ in (2-10) will become infinite, thus rendering the optimization problem meaningless.

(R2). Since x_o is, in general, not known exactly, it is impractical to "tune up" z_o to match x_o. Hence z_o is at rest [4], i.e. $z_o = 0$.

(R3). The compensator matrices P, N, G and H must be made independent of x_o; otherwise for every variation on x_o, a corresponding change in the compensator matrices is required. This can be achieved if it is supposed that [4] x_o is a random variable with *known* covariance, i.e., $E\{x_o\} = 0$, $E\{x_o x_o'\} \triangleq X_o$. (The zero-mean assumption is made for convenience of analysis only.) Then to remove the dependence of the compensator matrices on x_o, define a new performance index

$$J(F) = E\{\bar{J}(F)\} \qquad (2\text{-}11)$$

Clearly, J(F) so defined is an average measure of $\bar{J}$ for a given distribution of x_o.

On substitution of (2-10) into the last expression, noting the independence of z_o on x_o and remarks made in (R2), (R3),

$$J(F) = \tfrac{1}{2}\,\mathrm{tr}\left\{\int_0^{\infty} \Phi'(t,0)\,[Q + \overline{C}'F'\overline{R}F\overline{C}]\,\Phi(t,0)\,\overline{X}_o\,dt\right\} \tag{2-12}$$

where

$$\overline{X}_o \;\triangleq\; E\{\overline{x}_o\overline{x}_o'\} \;\equiv\; \begin{bmatrix} X_o & 0_{n,p} \\ 0_{p,n} & 0_{p,p} \end{bmatrix} \tag{2-13}$$

whence a new problem statement in place of (S1) may now be stated.

Problem statement (S2)

Given the plant (2-1) and the compensator (2-2) of given order p, $0 \leqslant p < n-r$, find the parameter matrix F such that the performance index (2-12) is minimized.

This is obviously a parameter optimization problem. The necessary conditions that must be satisfied by the optimal solution will be derived later in §2.2.3.

2.2.2 Canonical forms for P and N

The parameter matrix F defined in the compensator (2-2) has a total of $(p+m)\times(p+r)$ elements. However, in view of the fact that the compensator state z is arbitrary to within a (non-singular) linear transformation, p^2 of these elements are redundant. Thus, an upper bound on the number of independent parameters is given by

$$(p+m)\times(p+r) - p^2 \;=\; pm + pr + mr \tag{2-14}$$

In [4] and [10], the fact that there are redundancies in F has not been mentioned. It would be advantageous, although not essential, to adopt a canonical form for the compensator (2-2) that contains the reduced number of parameters (2-14).

This is in order to limit the dimensionality of the parameter optimization problem. Unfortunately, there is no such canonical form for MIMO systems that does not require some prior knowledge of the system.
For instance, if the structural indices of the optimal compensator (2-2) are known beforehand (of course they are not!), then the Bucy-Ackermann canonical form [8] may be used. The same problem arises in the identification of systems in state-variable form [30].

Notwithstanding these difficulties, it will now be shown that if the following (very mild) assumption is made, the elimination of $p^2 - p$ redundant parameters becomes possible.

Assumption

(A1). The optimal compensator matrix P is cyclic (this is even milder than assuming that the optimal P has distinct eigenvalues).

There is no further loss of generality in assuming P is in the companion form:

$$P = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ x & x & x & \cdots & x \end{bmatrix} \tag{C1}$$

If so desired, the remaining p redundant parameters may be eliminated by making the following stronger assumption.

Assumption

(A2). The optimal compensator (2-2) is completely controllable by at least one of its inputs y_i. Subsequently, by suitable numbering of the elements in y, it will be assumed that one such input is the first element of y.

Notice that assumption (A2) implies, but is not implied by, assumption (A1).

Then, with P in the form (C1), there is no further loss of generality in letting the first column of N be the constant vector (0, 0, ..., 0, 1)' i.e.,

$$N = \begin{bmatrix} 0 & x & \cdots & x \\ 0 & x & \cdots & x \\ \vdots & \vdots & & \vdots \\ 0 & x & \cdots & x \\ 1 & x & \cdots & x \end{bmatrix} \qquad \text{(C2)}$$

It may be verified that the canonical forms (C1) and (C2) result in a compensator that has the reduced number of parameters (2-14). However, as stated earlier, it is not essential that either of the foregoing assumptions (A1) and (A2) be made. These assumptions have the beneficial effect of reducing the dimensionality of the parameter optimization problem but at the risk of obtaining a less than optimal solution.

2.2.3 Main results

In this section, computable expressions for the performance index (J) and its gradient with respect to F will be derived. The following theorem is needed in the analysis. The proof of this theorem can be found in [9].

Theorem 1

Given the linear differential system

$$\mathring{x}(t) = M\,x(t), \quad x(t_o) = x_o \tag{T1-1}$$

Then,

$$\tfrac{1}{2}\int_{t_o}^{t_1} x'(t)\,R\,x(t)dt \equiv \tfrac{1}{2}\,x_o'\,K(t_o)x_o \tag{T1-2}$$

where $K(\cdot)$ is the symmetric, positive-definite matrix

$$K(t) = \int_{t}^{t_1} \Phi'(\tau,t)\,R\,\Phi(\tau,t)d\tau$$

$\Phi(t,t_o)$ is the transition matrix of system (T1-1). Also, $K(t)$ satisfies the matrix differential equation

$$-\mathring{K}(t) = M'K(t) + K(t)M + R \tag{T1-3}$$

with boundary condition

$$K(t_1) = 0 \tag{T1-4}$$

Corollary

If M is stable, then

$$\tfrac{1}{2}\int_{0}^{\infty} x'(t)R\,x(t)dt = \tfrac{1}{2}\,x_o'\,Kx_o \tag{T1-5}$$

and K satisfies the Lypunov equation

$$0 = M'K + KM + R \tag{T1-6}$$

Identify $\overline{A}_o$ in (2-3) with M in (T1-1) and $Q + \overline{C}'F'\overline{R}F\overline{C}$ in (2-12) with R in (T1-5), then

$$J(F) = \tfrac{1}{2}\,\mathrm{tr}\{K\,\overline{X}_o\} \tag{2-15}$$

where K satisfies the Lypunov equation

$$0 = \bar{A}_o'K + K\bar{A}_o + Q + \bar{C}'F'\bar{R}F\bar{C} \qquad (2\text{-}16)$$

Hence, the problem becomes one of minimizing J in (2-15) with respect to F subject to the constraint (2-16). From (2-15) and (2-16), form the Lagrangian L, thus

$$L = \tfrac{1}{2}\,\mathrm{tr}\{K\bar{X}_o\} + \tfrac{1}{2}\,\mathrm{tr}\{(\bar{A}_o'K + K\bar{A}_o + Q + \bar{C}'F'\bar{R}F\bar{C})L'\} \qquad (2\text{-}17)$$

where L is the $(n+p) \times (n+p)$ Lagrange multiplier matrix defined by

$$\frac{\partial L}{\partial K} = \bar{A}_oL + L\bar{A}_o' + \bar{X}_o \equiv 0 \qquad (2\text{-}18)$$

Then, the expression for the gradient of J with respect to F is

$$\frac{\partial J}{\partial F} = \bar{R}F\bar{C}L\bar{C}' + \bar{B}'KL\bar{C}' \qquad (2\text{-}19)$$

<u>Remarks</u>

(R4). Similar expressions have also been derived in [4], [10] for constant dynamic compensators using an extension of [3]. In [3], an application of Kleinman's lemma [11] to J so that a first order expansion of $e^{\bar{A}_ot}$ is obtained. The derivation is, however, much more involved than that shown above, which is similar to that proposed recently in [12].

(R5). Consider the design of constant gain controller (i.e., $p = 0$) and C is non-singular, e.g., $C = I_n$. Then it is readily seen that the results obtained above become the necessary and sufficient condition for the optimal linear state feedback solution derived by Kalman [13].

Equations (2-15), (2-16), (2-18) and (2-19) provide sufficient information for the minimization of J to be performed. To this end, several algorithms have been proposed, see e.g. [6]. These are discussed in the next section.

2.2.4 Solution algorithms

All the existing solution algorithms for the problem (S2) are iterative in nature. Each search is initialized with a guessed value for F such that (2-3) is closed-loop stable. After this, the procedure that is employed to obtain the optimal solution belongs to one of the following two types.

2.2.4.1 Indirect type

When (2-19) is set to zero, then

$$F = -\bar{R}^{-1}\bar{B}'KL\bar{C}'(\bar{C}L\bar{C}')^{-1} \tag{2-20}$$

which is a necessary condition for optimality. Any computational method which uses equation (2-20) in conjunction with (2-15), (2-16) and (2-18) is known as an *indirect-type algorithm*.

The first indirect-type computational algorithm is that suggested in [3]. After a stabilizing F has been chosen to initialize the search, K is then obtained from (2-16). Substitution of F (still unknown) from (2-20) into (2-18) with known K results in a *non-linear* equation with L as the only unknown, the solution of which is re-substituted into (2-20) to obtain a new F. This completes one iteration. The new value of F is then used to itialize the next iteration. It has been shown [3] that in this way J will decrease monotonically. However, to obtain the solution of the non-linear equation for L at every iteration is exceedingly time-consuming.

The method suggested in [6] is a variant of the above algorithm in that for a given F, K and L are solved using (2-16) and (2-18) respectively. These are then substituted into (2-20) to obtain a new F to complete an iteration. In this way, the solution of non-linear equation is avoided. Unfortunately, experience has shown that this indirect algorithm is computationally unsatisfactory for there always exists the risk of divergence.

2.2.4.2 Direct type

A new computational procedure of the direct-type will now be described. It was first proposed by Choi and Sirisena [7]. A similar approach can also be found in a later publication by Horisberger and Bélanger [14].

The essential difference between the proposed method in [7] and the indirect-type methods discussed earlier is that instead of setting (2-19) to zero as has been done in §2.2.4.1, the explicit expressions for J and $\partial J/\partial F$ are now used in a gradient-type function minimization technique such as that of Davidon-Fletcher-Powell (DFP) [15], see Appendix A. The optimal F is therefore determined by *direct* minimization of J. Of course, it is still necessary to prime the iterative search with a stabilizing F. The subsequent steps are therefore:

Step (i): Compute K, L using (2-16) and (2-18) respectively.

Step (ii): The cost J, gradient $\partial J/\partial F$ are obtained using (2-15) and (2-19) respectively.

Step (iii): Update F in accordance with the rules of the gradient algorithm used. Terminate search if $\|\delta F\|$ or $\|\partial J/\partial F\|$ is smaller than some prescribed tolerance. Otherwise return to step (i).

Remarks

(R6). Notice that for fixed F, the computation of J and $\partial J/\partial F$ only requires the solution of linear equations (2-16) and (2-18). Several methods are available to solve equations of this type, see e.g. [16]-[18].

(R7). All values of F tried during the subsequent unidirectional searches in step (iii) above must also stabilize the closed-loop system (2-3). Now, because of the positiveness of the integrand in (2-12), it is evident that $J \to \infty$ as F approaches the stability boundary. Hence, in view of the fact that the derivative of J exists, a minimum of J must occur *before* the stability boundary is encountered. There remains the possibility that too large a step taken during the initial stages of the search could take F into the unstable region. However, this can be avoided by subjecting the value of F to a Hurwitz stability test (see e.g. [19]) although computational evidence indicates that such a precaution is not usually necessary.

(R8). If the canonical forms (C1) and (C2) in §2.2.2 are assumed, then the derivatives of J with respect to the independent parameters (the crosses) are picked out of the gradient matrix ($\partial J/\partial F$) for use in the iterative search. The components of $\partial J/\partial F$ corresponding to the fixed compensator parameters (the ones or zeros) may be disregarded. In this way, the total number of parameters which need to be considered in the minimization procedure is equal to pm + mr + pr. The computational efficiency is therefore enhanced.

(R9). It is conceivable that several local optimal solutions may exist because J is, in general, a non-convex function of F. Hence, irrespective of what algorithm is employed, iterations should be commenced for different priming values of F.

2.2.5 Numerical example

The numerical example is taken from [20] and refers to a Mach 2.7 flight condition of a supersonic transport aircraft. The system equations are

$$\dot{x} = \begin{bmatrix} -.037 & .0123 & .00055 & -1.0 \\ 0. & 0. & 1.0 & 0. \\ -6.37 & 0. & -.23 & .0618 \\ 1.25 & 0. & .016 & -.0457 \end{bmatrix} x + \begin{bmatrix} .00084 & .000236 \\ 0. & 0. \\ .08 & .804 \\ -.0862 & -.0665 \end{bmatrix} u$$

Suppose a constant gain feedback controller (p = 0) is to be constructed using only the last three state variables, i.e. $C = [0_{3,1} \quad I_3]$. For $Q = I_4$, $R = I_2$ and $X_o = I_4$, the feedback gains converged after 35 iterations of the DFP algorithm to

$$F = \begin{bmatrix} -.36 & -1.53 & -7.61 \\ 1.27 & 3.54 & 5.06 \end{bmatrix}$$

with the corresponding cost $J = 79.56$. This compares with the priming values $F = 0_{2,3}$ and $J = 15568$. The convergence criterion was

$$\sum_{i=1}^{6} \left(\frac{\partial J}{\partial F_i}\right)^2 < .1 \quad .$$

2.3 OPTIMAL OUTPUT FEEDBACK CONTROLS FOR UNSTABLE PLANTS

In this section, the design of constant-gain output feedback controllers for unstable plants is considered. The extension of the results to the design of dynamic compensators of the form (2-3) is straightforward. The results obtained in this section first appeared in a paper by Choi and Sirisena [31].

Problem statement (S3)

Given the plant

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_o \\ y(t) &= Cx(t) \end{aligned} \tag{2-21}$$

where $x(t) \varepsilon R^n$, $u(t) \varepsilon R^m$ and $y(t) \varepsilon R^r$, find the constant matrix such that

$$u(t) = Fy(t) \tag{2-22}$$

minimizes the performance index

$$J = \tfrac{1}{2}E\{\int_0^\infty x'(t)Qx(t) + u'(t)Ru(t)dt\}, \quad Q \geqslant 0, \quad R > 0 \tag{2-23}$$

It is assumed as in (R3) that x_o is a random variable with known covariance, $E\{x_o\} = 0$, $E\{x_o x_o'\} = X_o$.

As has been pointed out earlier, all the iterative techniques described in §2.2.4 require an initial guess of F that stabilizes the closed-loop system. If (2-21) is open-loop stable, then the choice $F = 0_{m,r}$ suffices. However, the choice becomes difficult when the open-loop system is unstable; one may have to resort to an auxiliary stabilization algorithm such as those proposed in [21], [22] or Chapter 4 [23].

However, an algorithm which is an extension of the direct method described in §2.2.4.2 will be described shortly that requires only a stabilizing *state* feedback law to initialize the search. Such a law always exists for a controllable system, and moreover can be easily found using, for example, the method of Kleinman [24].

2.3.1 Development

Without loss of generality, it is assumed that

$$C = [I_r \quad 0_{r,n-r}] \tag{2-24}$$

and define

$$\bar{C} \triangleq [0_{n-r,r} \quad I_{n-r}] \tag{2-25}$$

Now define a general linear state feedback law

$$u = \hat{F}x \triangleq [F \quad \bar{F}] \left[\begin{array}{c} C \\ \hline \bar{C} \end{array}\right] x \tag{2-26}$$

Clearly, F corresponds to the $m \times r$ feedback matrix from y while the $m \times (n-r)$ submatrix $\bar{F}$ corresponds to the feedback from the inaccessible system states. Hence, the problem of finding the optimal output feedback law is equivalent to that of minimizing (2-23) with respect to $\hat{F}$ subject to the constraint $\bar{F} = 0_{m,n-r}$. This is a constrained optimization problem that may be solved numerically using standard techniques.

One such technique that has appeared to have had considerable success is the method of multiplier [25]. To apply this method, define a scalar penalty weight $\gamma(\geq 0)$ and a matrix multiplier Λ and form an augmented performance index

$$\hat{J} = J + \mathrm{tr}\{\bar{F}\Lambda' + \tfrac{1}{2}\gamma\bar{F}\bar{F}'\} \qquad (2\text{-}27)$$

Each cycle of the method consists of the (unconstrained) minimization of $\hat{J}$ with respect to $\hat{F}$ followed by updating of the multiplier Λ according to an appropriate rule. This procedure eventually converges to the constrained optimal solution.

The unconstrained minimization is perhaps best performed using a gradient-type algorithm, such as that of DFP which is also used in §2.2.4.2. This would require computable expressions for $\hat{J}$ and its gradient with respect to $\hat{F}$, and these will now be derived.

From the results shown in §2.2.3 and recognising that $\begin{bmatrix} C \\ \hline \bar{C} \end{bmatrix} \equiv I_n$,

$$J = \tfrac{1}{2}\,\mathrm{tr}[(Q + \hat{F}'R\hat{F})X_o] \qquad (2\text{-}28)$$

$$\frac{\partial J}{\partial \hat{F}} = R\hat{F}K + B'LK \qquad (2\text{-}29)$$

where K, L are solutions of the matrix equations

$$(A + B\hat{F})K + K(A + B\hat{F})' + X_o = 0 \qquad (2\text{-}30)$$

$$(A + B\hat{F})'L + L(A + B\hat{F}) + Q + \hat{F}'R\hat{F} = 0 \qquad (2\text{-}31)$$

Therefore, $\hat{J}$ can now be evaluated from (2-27). Moreover,

$$\frac{\partial \hat{J}}{\partial \hat{F}} = \frac{\partial J}{\partial \hat{F}} + [0_{m,r} \vdots \Lambda + \gamma \bar{F}] \tag{2-32}$$

which clearly is also computable.

2.3.2 The algorithm

The complete procedure for determining the optimal output feedback law is as follows:-

Step (i): Set $\Lambda = 0_{m,n-r}$ and γ to a moderately large (arbitrary) value. (Theoretical considerations [26] show that γ must be sufficiently large for the function $\hat{J}$ to be convex.)

Step (ii): Choose a stabilizing state feedback law $\hat{F}$ to initialize the search.

Step (iii): Minimize $\hat{J}$ with respect to $\hat{F}$ using the DFP algorithm, with a limit on the search step to ensure that $\hat{F}$ is always stabilizing as discussed in remark (R7). Proceed to step (iv) when $\|\partial\hat{J}/\partial\hat{F}\|$ is less than a prescribed tolerance.

Step (iv): If $\|\bar{F}\|$ is less than some prescribed tolerance, then F is an optimal feedback law and the search is terminated. Otherwise, update the multiplier according to

$$\Lambda_{i+1} = \Lambda_i + \gamma \bar{F}_i \tag{2-33}$$

where the subscript i denotes values at the end of the i^{th} cycle, and return to step (iii) to commence a new cycle.

Remarks

In addition to the remarks made in (R7) and (R9) the following remarks also apply.

(R10). Failure to make $\overline{F}$ sufficiently small for a variety of starting $\hat{F}$ would indicate that the plant (2-21) cannot be stabilized by constant output feedback. It would then become necessary to try feedback via a dynamic compensator. Since it has been shown in §2.2.1 that the composite plant-compensator system can be described by equations of the form (2-21) and (2-22), the procedure outlined above can also be employed (with only minor modifications) to find the optimal compensator parameters.

(R11). The updating formula (2-33) is the one suggested by Hestenes [25]. However, other updating rules exist in the literature, see e.g. [27].

(R12). The above procedure may also be employed as a stabilizing algorithm; the performance index (2-23) in this case can be chosen arbitrarily as long as it is positive definite.

2.3.3 Numerical example

The feasibility of the proposed technique will now be demonstrated by means of a numerical example, the data for which is taken from [28].

The lateral motion of an aircraft is represented by the state equation

$$\dot{x}(t) = \begin{bmatrix} -.154 & .004 & -.99 & .178 & .075 \\ -1.25 & -2.85 & 1.43 & 0. & -.727 \\ .568 & -.277 & -.284 & 0. & -2.05 \\ 0. & 1. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & -10. \end{bmatrix} x(t) + \begin{bmatrix} 0. \\ 0. \\ 0. \\ 0. \\ 10. \end{bmatrix} u(t)$$

Assuming that only the first three states are available for feedback, i.e. $C = [I_3 \quad 0_{3,2}]$, the problem is to find the feedback law (2-22) that minimizes the performance index (2-23) with $Q = I_5$, $R = I_1$ and $X_o = I_5$.

The given plant has an unstable pole at .033, but it is known from [28] that the system can be stabilized using feedback from only the first four state variables. Hence when using the procedure developed in this section, the feedback gain from x_5 may be set to zero throughout the computation. The number of free parameters in $\hat{F}$ is thereby reduced from five to four with a corresponding reduction in solution time.

With the stabilizing control law

$$u(t) = [-.976 \quad -.0054 \quad .848 \quad .175 \quad 0.]\; x(t)$$

given in [28] being used to initialize the search, the results obtained were as follows:

$$F_{opt} = [.127 \quad .788 \quad 1.215]$$

with

$$J_{min} = 5.6869$$

and

$$\|\bar{F}\| \triangleq [tr(\bar{F}\bar{F}')]^{\frac{1}{2}} < 10^{-4} .$$

The corresponding closed-loop poles are -.212, -.291, -6.434, -3.156 ± j2.052. In contrast, the feedback obtained in [28] via a least-square pole placement technique using feedback from the same three states has an unstable closed-loop pole.

2.4 CONCLUSION

Optimal output feedback control of LMS for quadratic performance index has been considered in this chapter. It has been shown how the parameter optimization problem can be solved readily using a gradient technique. Unlike other algorithms, the proposed algorithm avoids the solution of non-linear matrix equations while appearing to ensure convergence, thus making this approach to the design of optimal feedback control systems more viable. The gradient technique will also be used in Chapter 3 which deals with the optimal output feedback control of stochastic systems. Consideration was also given in this chapter to the control of open-loop unstable plant. A modified gradient-technique was developed where the difficulty of choosing a stabilizing output feedback law to initialize the iterative design procedure has been avoided. The method only requires a stabilizing *state* feedback law which can be easily obtained using standard techniques.

APPENDIX A

Davidon-Fletcher-Powell function minimization subroutine (FMFP)

The subroutine FMFP described in an IBM publication (System/360 Scientific Subroutine Package (360A-CM-63X), Version III, page 221) has been used extensively by the author for function minimization. Unfortunately, it has been discovered that there is a serious software error in the original program and hence the following modifications were made.

Adopting the notation used in FMFP,

(M1) Increase the dimension of vector H from N(N + 7)/2 to N(N + 9)/2.

(M2) Insert between line FMFP830 and line FMFP840, the statement

```
N4 = N*(N + 7)/2
```

(M3) Insert between line FMFP990 and line FMFP1000, the statements

```
IKAN    = N4 + J
H(IKAN) = X(J)
```

(M4) Change line FMFP2960 to

```
K = N4 + J
```

The rest of FMFP remains unchanged.

REFERENCES

[1] Rosenbrock, H.H., "Progress in the design of multivariable control systems," Measurement and Control, vol. 4, pp.9-11, Jan. 1971.

[2] MacFarlane, A.G.J., "A survey of some recent results in linear multivariable feedback theory," Automatica, vol. 8, pp.455-492, 1972.

[3] Levine, W.S. and Athans, M., "On the determination of the optimal constant output feedback gains for linear multivariable systems," IEEE Trans. Automat. Contr., vol. AC-15, pp.44-48, Feb. 1970.

[4] Johnson, T.L. and Athans, M., "On the design of optimal constrained compensators for linear control systems," IEEE Trans. Automat. Contr., vol. AC-15, pp.658-660, Dec. 1970.

[5] Newton, G.C., Jr., Gould, L.A. and Kaiser, J.F., Analytical Design of Linear Feedback Controls. New York: Wiley, 1957.

[6] Anderson, B.D.O. and Moore, J.B., Linear Optimal Control. Englewood Cliffs, N.J.: Prentice Hall, 1971.

[7] Choi, S.S. and Sirisena, H.R., "Computation of optimal output feedback gains for linear multivariable systems," IEEE Trans. Automat. Contr., vol. AC-19, pp.257-258, June 1974.

[8] Ackermann, J.E. and Bucy, R.S., "Canonical minimal realization of a matrix of impulse response sequences," Information and control, vol. 19, pp.224-231, 1971.

[9] Kwakernaak, H. and Sivan, R., Linear Optimal Control Systems. New York: Wiley-Interscience, 1972.

[10] Levine, W.S., Johnson, J.L. and Athans, M., "Optimal limited stat variable feedback controllers for linear systems," IEEE Trans. Automat. Contr., vol. AC-16, pp.785-743, Dec. 1971.

[11] Kleinman, D.L., "On the linear regulator problem and the matrix Riccati equation," M.I.T. Electronic Systems Lab., Cambridge, Mass., Tech. Rept. ESL-R-271, 1966.

[12] Mendel, J.M., "A concise derivation of optimal constant limited state feedback gains," IEEE Trans. Automat. Contr., vol. AC-19, pp.447-448, Aug. 1974.

[13] Kalman, R.E., "Contributions to the theory of optimal control," Bol. Soc. Mat. Mex., Second Series, vol. 5, pp.102-119, 1960.

[14] Horisberger, H.P. and Bélanger, P.R., "Solution of the optimal constant output feedback problem by conjugate gradients," IEEE Trans. Automat. Contr., vol. AC-19, pp.434-435, Aug. 1974.

[15] Fletcher, R. and Powell, M.J.D., "A rapidly convergent descent method for minimization," British Computer J., pp.163-168, June 1963.

[16] Chen, C.F., "A note on expanding PA + A'P = -Q," IEEE Trans. Automat. Contr., vol. AC-13, pp.122-123, Feb. 1968.

[17] Davison, E.J. and Man, F.T., "The numerical solution of A'Q + QA = -C," IEEE Trans. Automat. Contr., vol. AC-13, pp.449, June 1968.

[18] Molinari, B.P., "Algebraic solution of matrix linear equations in control theory," Proc. Inst. Elec. Eng., vol. 116, pp.1748-1754, Oct. 1969.

[19] Shinners, S.M., Control Systems Designs. New York, Wiley, 1964.

[20] Markland, C.A., "Optimal model-following control system synthesis techniques," Proc. Inst. Elec. Eng., vol. 117, pp.623-627, 1970.

[21] Fortmann, T.E., "Stabilization of multivariable systems with constant-gain output feedback," in Proc. JACC, paper 10-6, 1973.

[22] Kreisselmeier, G., "Stabilization of linear systems by constant output feedback using the Riccati equation," IEEE Trans. Automat. Contr., vol. AC-20, pp.556-557, Aug. 1975.

[23] Sirisena, H.R. and Choi, S.S., "Pole placement in prescribed regions of the complex plane using output feedback," IEEE Trans. Automat. Contr., vol. AC-20, pp.810-812, Dec. 1975.

[24] Kleinman, D.L., "An easy way to stabilize a linear constant system," IEEE Trans. Automat. Contr., vol. AC-15, pp.692, Dec. 1970.

[25] Hestenes, M.R., "Multiplier and gradient methods," J. Optimiz. Theory Appl., vol. 4, no. 5, pp.303-320, 1969.

[26] Bertsekas, D.P., "Combined primal-dual and penalty methods for constrained minimization," SIAM J. Contr., vol. 13, no. 3, pp.521-542, May 1975.

[27] Miele, A., Moseley, P.E., Levy, A.V. and Coggins, G.M., "On the method of multipliers for mathematical programming problems," J. Optimiz. Theory Appl., vol. 10, no. 1, pp.1-33, 1972.

[28] Jameson, A., "Design of a single-input system for specified roots using output feedback," IEEE Trans. Automat. Contr., vol. AC-15, pp.345-348, June 1970.

[29] Luenberger, D.G., "Observers for multivariable systems," IEEE Trans. Automat. Contr., vol. AC-11, pp.190-197, April 1966.

[30] Denham, M.J., "Canonical forms for the identification of multivariable linear systems," IEEE Trans. Automat. Contr., vol. AC-19, pp.646-656, Dec. 1974.

[31] Choi, S.S. and Sirisena, H.R., "Computation of optimal output feedback controls for unstable linear multivariable systems," IEEE Trans. Automat. Contr., accepted for publication.

CHAPTER 3

DESIGN OF OUTPUT FEEDBACK CONTROLLERS FOR STATIONARY AND NON-STATIONARY STOCHASTIC SYSTEMS USING QUADRATIC PERFORMANCE INDICES

3.1 INTRODUCTION

Chapter 2 dealt with the design of optimal output feedback controllers for essentially deterministic MIMO regulator systems. In this chapter, the design technique is extended to systems with significant amounts of measurement and process noise; both stationary and non-stationary stochastic systems are considered.

A significant theoretical development in the linear stochastic optimal control problem is the *separation theorem* [1]. This states that the (unconstrained) optimal control scheme consists of linear feedback from the minimum variance state estimates provided by a Kalman filter, the feedback gains being precisely equal to those for the corresponding deterministic (noise-free) problem. Such a control scheme would, however, be exceedingly complex and hence impractical in most applications due to the high-order filter required. This difficulty may be avoided by employing lower-order fixed-configuration dynamic compensators like those adopted in Chapter 2.

Two practical cases can be distinguished:

(i) The optimal steady-state control of time-invariant systems subject to random disturbances having time-invariant statistics, i.e., the stationary stochastic optimal control problem.

(ii) The optimal control of possibly time-varying systems over a finite time interval where the noise-statistics may also be time-varying, i.e., the non-stationary stochastic optimal control problem.

<u>Stationary stochastic output feedback control problem</u>

The design of low-order, fixed-configuration compensators for stationary stochastic systems is considered in §3.2. Both measurement and process noise are allowed for in the problem formulation; moreover, measurement noise need not necessarily be white. Again, computable expressions are obtained for the cost and its gradients with respect to the compensator parameters to enable the optimal parameters to be determined by a direct gradient-type algorithm. It is interesting to note that the results are not unlike those obtained for the deterministic output feedback problem considered in Chapter 2.

Some of these results were first presented in a paper by Sirisena and Choi [2]. Similar results have also been obtained in several later publications, see e.g. [3] and [4]. Recently, Kurtaran [5] has considered the corresponding discrete-time version of the problem.

The problem treated by the aforementioned researchers can, of course, be solved using classical parameter optimization techniques [6] but at the expense of tedious hand calculations. The algorithm of Åström [7] for evaluating the special integral that arises in this earlier approach, though helpful, does not completely obviate these calculations.

Non-stationary stochastic output feedback control problem

Problems of this class were first treated by Axsäter [8] who determined the optimal time-varying gains in the presence of plant input noise but assuming noise-free measurements. Sidar and Kurtaran [9] have extended Axsäter's results to include the design of dynamic compensators. They also consider the problem of coloured measurement noise which is mathematically equivalent to the noise-free measurement problem. Sims and Melsa [10] attempted to extend the treatment to include measurements containing white noise; however, their indirect approach, based on the solution of non-linear two-point boundary-value problem, failed to yield the solution because the problem was then singular in some of the compensator parameters.

§3.3 of the present chapter contains a successful treatment of problems with both white and coloured measurement noise. Here the solution is approached through direct gradient-type methods which can handle singular problems. The new algorithm can, of course, also be used to solve the Axsäter problem; moreover, since it is of the direct-type, it should possess better global convergence properties than the indirect-type algorithm of Axsäter.

The implementation of the strictly time-varying compensators may present practical difficulties. Hence, consideration is also given to further simplification of the compensator structure by constraining its gains to be weighted sums of prechosen functions of time, as in [12], or piecewise-constant functions of time, as in [13].

The results of § 3.3 were first presented in a paper by Sirisena and Choi [14]. The special case of constant compensator gains has also been treated by Basuthakur and Knapp [4] in a paper that appeared after the present work was completed.

3.2 STATIONARY STOCHASTIC REGULATOR PROBLEM

3.2.1 Problem formulation

The closed-loop system under consideration is depicted in Fig. 3.1.

The plant is governed by the differential equations

$$\dot{x}(t) = A\,x(t) + B\,u(t) + w(t) \tag{3-1}$$

where $x(t) \varepsilon R^n$, $u(t) \varepsilon R^m$ are the plant state and control vectors respectively, $w(t) \varepsilon R^n$ is a white noise process with

$$E\{w(t)\} = 0, \qquad E\{w(t)w'(\tau)\} = W\delta(t-\tau)$$

The measurement vectors are

$$\left.\begin{aligned} y_1(t) &= C_1\,x(t) + v_1(t) \\ y_2(t) &= C_2\,x(t) \end{aligned}\right\} \tag{3-2}$$

where $y_1(t)$ is an r_1-vector of noisy measurements, while $y_2(t)$ is an r_2-vector of noise-free measurements. Also, $v_1(t)$ is an r_1-dimensional vector white noise process with

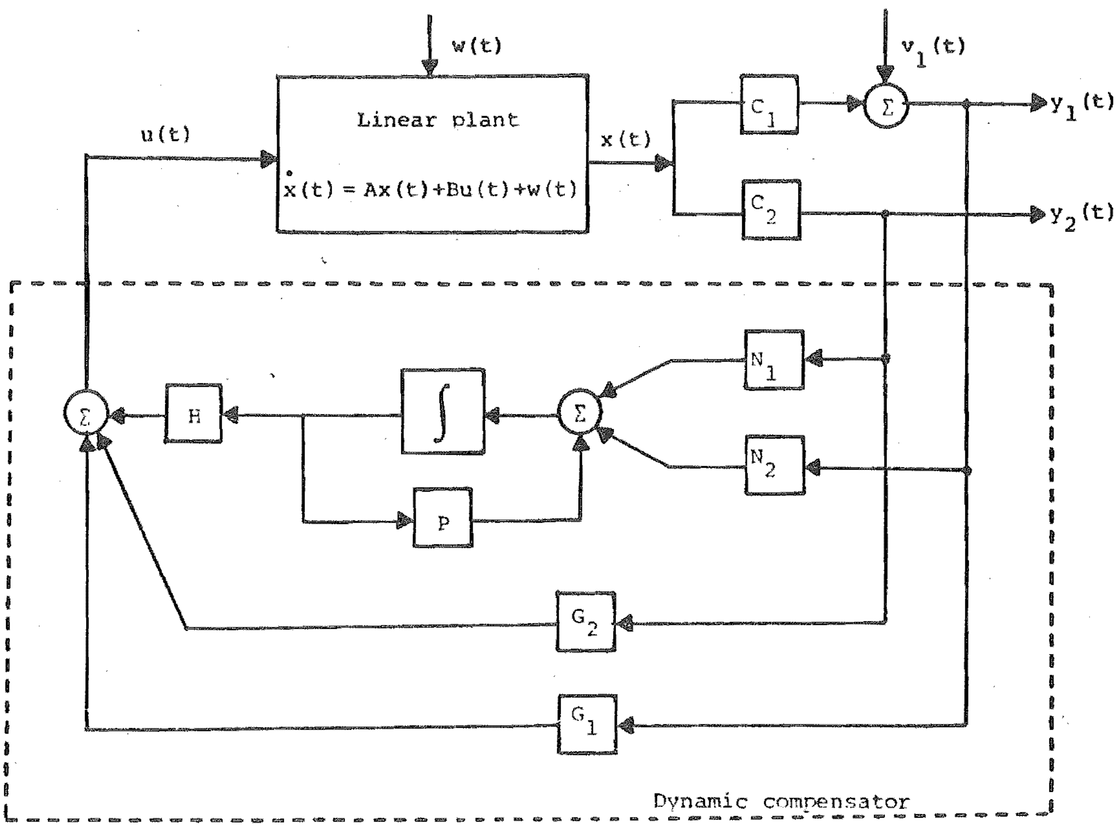

Fig. 3.1 The closed-loop stochastic system

$$E\{v_1(t)\} = 0, \quad E\{v_1(t)v_1'(\tau)\} = V_1\delta(t-\tau)$$

where the covariance matrix V_1 is positive definite. Also, $v_1(t)$ and $w(t)$ are assumed to be uncorrelated.

Remark

(R1). Instead of (3-2), suppose some of the measurements are contaminated with coloured noise (σ) having a rational power spectral density. The coloured noise process may be modelled by the differential equation

$$\dot{\sigma}(t) = A_\sigma \sigma(t) + v_2(t)$$

where $v_2(t)$ is a white-noise source. When σ is augmented to the state vector $x(t)$, it can be easily verified that the composite state and measurement equations are in the forms (3-1) and (3-2). Hence, the case of coloured measurement noise is allowed for in the above formulation.

The compensator to be designed is of the form

$$\left.\begin{aligned} \dot{z}(t) &= Pz(t) + N_1y_1(t) + N_2y_2(t) \\ u(t) &= Hz(t) + G_1y_1(t) + G_2y_2(t) \end{aligned}\right\} \qquad (3\text{-}3)$$

where $z(t)\varepsilon R^p$ is the compensator state vector. P, N_1, N_2, H, G_1, G_2 are constant matrices that remain to be determined. p is fixed a priori.

Equations (3-1) and (3-3) can be combined to form a composite closed-loop system.

$$\dot{\bar{x}}(t) = (\bar{A} + \bar{B}F\bar{C})\bar{x}(t) + \bar{B}F\bar{v}(t) + \bar{w}(t) \qquad (3\text{-}4)$$

where

$$\bar{x} \triangleq \begin{bmatrix} x \\ z \end{bmatrix}, \quad \bar{v}(t) \triangleq \begin{bmatrix} v_1(t) \\ O_{r_2+p,1} \end{bmatrix}, \quad \bar{w}(t) \triangleq \begin{bmatrix} w(t) \\ O_{p,1} \end{bmatrix},$$

$$\bar{A} \triangleq \begin{bmatrix} A & O_{n,p} \\ O_{p,n} & O_{p,p} \end{bmatrix}, \quad \bar{B} \triangleq \begin{bmatrix} B & O_{n,p} \\ O_{p,m} & I_p \end{bmatrix}, \quad \bar{C} \triangleq \begin{bmatrix} C_1 & O_{r_1,p} \\ C_2 & O_{r_2,p} \\ O_{p,n} & I_p \end{bmatrix}, \tag{3-5}$$

$$F \triangleq \begin{bmatrix} G_1 & G_2 & H \\ N_1 & N_2 & P \end{bmatrix}.$$

The compensator matrix F is to be chosen so as to minimise the quadratic performance index

$$J \triangleq \lim_{t\to\infty} \tfrac{1}{2} E\{x'Qx + u'Ru\} \tag{3-6}$$

with $Q \geq 0$, $R > 0$. Since system noise is being allowed for explicitly, it is not necessary to weight the compensator gains, as has been done in Chapter 2 for the deterministic problem, in order to limit noise transmission. (3-6) is actually a generalization of the classical mean-square-error criterion for stationary process [6]. Clearly, an optimal F exists only if the triple $(\bar{A}, \bar{B}, \bar{C})$ is stabilizable.

<u>Remark</u>

(R2). Other forms of quadratic performance indices also appear in the literature, see e.g. [4]. Such performance indices may also be handled by the techniques developed below.

Since white noise has an infinite variance, the performance index (3-6) will be finite only if the control input u(t) is free of white noise, and this is true only if

$$G_1 \equiv 0_{m,r_1} \tag{3-7a}$$

Hence, in what follows, it is assumed that (3-7a) is always satisfied. Also, in view of (3-7a), the problem has the trivial solution

$$u(t) = 0$$

unless at least one of the following conditions is satisfied:

$$\left.\begin{aligned} r_2 &\geqslant 1 \\ p &\geqslant 1 \end{aligned}\right\} \tag{3-7b}$$

Thus a non-trivial solution is possible for memoryless output feedback (i.e., p = 0) only if there is at least one noise-free observation (or if the observation noise is coloured). However, if $p \geqslant 1$, non-trivial solutions may exist if all the observations are contaminated by white noise.

<u>Remark</u>

(R3). The optimal compensator designed using the present approach can be related to the optimal controller obtained via the Bryson-Johansen [11] filter approach in the following situation. Consider the case when every element of $\dot{y}_2(t)$ either contains white noise or is already included in $y_2(t)$, the optimal compensator then consists of a differentiatorless Bryson-Johansen filter (order $n-r_2$) followed by a memoryless linear transformation. The structure of the optimal compensator is, in this case, identical to that shown in (3-3) and hence, there is no advantage to be gained by increasing p beyond $n-r_2$ in the present design approach.

However, for the contrary case, the present approach could only yield optimal (fixed-configuration) differentiatorless compensators because, in general, the Bryson-Johansen filter contains differentiators. It is now conceivable that the closed-loop system performance will continue to improve as p increases beyond $n-r_2$; however, this point has yet to be satisfactorily resolved. Of course, the present approach will yield the optimal unconstrained compensator if $y_2(t)$ is augmented beforehand with all the time derivatives of its elements that do not contain white noise.

The optimization problem may now be stated.

Problem Statement (S1)

For the stationary stochastic system (3-4), find F such that the performance index (3-6) is minimized; the submatrix G_1 being constrained to be a null matrix.

As has been pointed out in § 2.2.2, since the compensator state z(t) is arbitrary within a (non-singular) linear transformation, there are p^2 redundant parameters in F. The canonical forms (C1) and (C2) may be adopted to remove this redundancy, but possibly at the risk of obtaining a less than optimal solution. (See detailed discussion in § 2.2.2.)

3.2.2 Solution technique

Computable expression for J(F) will now be derived. Substitution of (3-3) into (3-6), and simplifying the resulting expression using (3-5), yields

$$J = \lim_{t\to\infty} \tfrac{1}{2} E\{\overline{x}'(\overline{Q}+\overline{C}'F'\overline{R}F\overline{C})\overline{x}\} \tag{3-8}$$

where

$$\overline{Q} = \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}, \qquad \overline{R} = \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} \tag{3-9}$$

Now let $\overline{X} \triangleq \lim_{t\to\infty} E\{\overline{x}(t)\overline{x}'(t)\}$. Then it may be easily shown [19] that $\overline{X}$ satisfies the matrix equation

$$(\overline{A}+\overline{B}F\overline{C})\overline{X} + \overline{X}(\overline{A}+\overline{B}F\overline{C})' + \overline{W} + \overline{B}F\overline{V}F'\overline{B}' = 0 \tag{3-10}$$

where

$$\overline{W} \triangleq \begin{bmatrix} W & 0 \\ 0 & 0 \end{bmatrix}; \qquad \overline{V} \triangleq \begin{bmatrix} V_1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{3-11}$$

Consequently, from (3-8)

$$J = \tfrac{1}{2} \operatorname{tr}\{(\overline{Q} + \overline{C}'F'\overline{R}F\overline{C})\overline{X}\} \tag{3-12}$$

The gradient $\partial J/\partial F$ may now be obtained via the matrix calculus method shown in §2.2.3. Form the Lagrangian $\mathcal{L}$ by appending the side-constraint (3-10) on J(F), thus

$$\mathcal{L} = \tfrac{1}{2}\operatorname{tr}\{(\overline{Q}+\overline{C}'F'\overline{R}F\overline{C})\overline{X}\} + \tfrac{1}{2}\operatorname{tr}\{[(\overline{A}+\overline{B}F\overline{C})\overline{X}+\overline{X}(\overline{A}+\overline{B}F\overline{C})' + \overline{W}+\overline{B}F\overline{V}F'\overline{B}']L'\} \tag{3-13}$$

Where the $(n+p)\times(n+p)$ Lagrange multiplier matrix L also satisfies the matrix equation

$$\frac{\partial\mathcal{L}}{\partial\overline{X}} = (\overline{A}+\overline{B}F\overline{C})'L + L(\overline{A}+\overline{B}F\overline{C}) + \overline{Q} + \overline{C}'F'\overline{R}F\overline{C} = 0 \tag{3-14}$$

Lastly, the gradient $\partial J/\partial F$ is given by

$$\frac{\partial \mathcal{L}}{\partial F} = \frac{\partial J}{\partial F} = \bar{R}F\bar{C}X\bar{C}' + \bar{B}'LX\bar{C}' + \bar{B}'L\bar{B}F\bar{V} \qquad (3\text{-}15)$$

Since computable expressions for J and $\partial J/\partial F$ have been obtained, the direct-type algorithm described in §2.2.4.2 can be used (with only slight modifications) for the minimization of J(F). Also, remarks (R7)-(R9) of Chapter 2 concerning system stability and solution uniqueness are also applicable in this problem.

3.3 NON-STATIONARY STOCHASTIC REGULATOR PROBLEM

Attention will now be directed towards the optimal output feedback control of non-stationary stochastic systems.

3.3.1 The problem

The time-varying plant considered is described by the state equation

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + w(t) \qquad (3\text{-}16)$$

where x(t) and u(t) are as defined in § 3.2.1. The n-dimensional vector white noise process w(t) is, however, non-stationary with

$$E\{w(t)\} = 0, \qquad E\{w(t)w'(\tau)\} = W(t)\delta(t-\tau).$$

The initial plant state $x(t_o)$ is a random variable with

$$E\{x(t_o)\} = \mu_o, \qquad E\{x(t_o)x'(t_o)\} = X_o.$$

The assumption on the observation vectors made in § 3.2.1 is also used here, i.e.

$$\left.\begin{aligned} y_1(t) &= C_1(t)x(t) + v_1(t) \\ y_2(t) &= C_2(t)x(t) \end{aligned}\right\} \qquad (3\text{-}17)$$

where $y_1(t)$ is an r_1-vector of noisy observations, $y_2(t)$ is an r_2-vector of noise-free observations, and $v_1(t)$ is an r_1-dimensional vector white noise process with

$$E\{v_1(t)\} = 0, \qquad E\{v_1(t)v_1'(\tau)\} = V_1(t)\delta(t-\tau),$$

where the covariance matrix $V_1(t)$ is positive definite. Also, $x(t_o)$, $v(t)$ and $w(t)$ are assumed to be uncorrelated. Clearly, as has been pointed out in (R1), § 3.2.1, coloured observation noise is allowed for in the formulation.

The problem is to find the fixed-order dynamic compensator of the form

$$\left.\begin{aligned} \dot{z}(t) &= P(t)z(t) + N_1(t)y_1(t) + N_2(t)y_2(t) \\ u(t) &= H(t)z(t) + G_1(t)y_1(t) + G_2(t)y_2(t) \end{aligned}\right\} \qquad (3\text{-}18)$$

where $z(t)$ is the p-dimensional compensator state, that minimizes the performance index

$$J = E\{x'(t_f)Sx(t_f) + \int_{t_o}^{t_f} [x'(t)Q(t)x(t) + u'(t)R(t)u(t)]dt\}, \qquad (3\text{-}19)$$

where S and $Q(t)$ are symmetric positive semidefinite matrices, and $R(t)$ is a symmetric positive definite matrix. Notice that in (3-19) the generalized mean-square-error $x'(t)Q(t)x(t) + u'(t)R(t)u(t)$ is integrated over a finite time interval because the system considered here is non-stationary. Also, some control in the magnitude of $x(t)$ at the final time (t_f) is often desired. This explains the appearance of the term $x'(t_f)Sx(t_f)$ on the RHS of (3-19). Finally, for reasons stated earlier in §3.2.1, no weighting on the compensator gains is included in J and conditions (3-7a) and (3-7b) must also be satisfied in the present problem, i.e.

$$G_1(t) = 0 \tag{3-20}$$

and either

$$\left.\begin{aligned} & r_2 \geqslant 1 \\ \text{and/or} \quad & p \geqslant 1 \end{aligned}\right\} \tag{3-21}$$

Although $z(t_o)$ may be assumed to be an arbitrary constant, as in Chapter 2, somewhat better system performance may be possible if all the a priori information is utilized. Hence, $z(t_o)$ is assumed to be a linear combination of the expected value of the initial state $x(t_o)$ and the noise-free initial observation $y_2(t_o)$, i.e.

$$z(t_o) = \alpha\mu_o + \beta y_2(t_o) \tag{3-22}$$

where α and β are constant matrices to be chosen so as to minimize (3-19).** Notice that (3-22) is of the same form as the corresponding relation for the Bryson-Johansen filter. Since μ_o is known, the number of problem variables may be reduced by writing (3-22) in the form

$$z(t_o) = \gamma_o + \beta y_2(t_o) \tag{3-23}$$

**Note that Sidar and Kurtaran [9] have not considered the term $\beta y_2(t_o)$.

The problem thus is to find the compensator matrices $P(t)$, $H(t)$, $N_1(t)$, $N_2(t)$ and $G_2(t)$, the vector γ_o and the matrix β that minimize the performance index (3-19).

Since the terminal time is finite, the performance index will always be finite provided (3-20) is satisfied. Hence, it is not necessary that there exist a stabilising compensator of the form (3-18) for the problem to have a solution. Finally, the conclusion drawn in (R2) concerning compensator order is also valid in the present formulation.

3.3.2 Solution technique

It may be verified easily that the (n+p)-th order closed-loop system defined by (3-16), (3-17) and (3-18) may be described by the composite state equation

$$\dot{\bar{x}}(t) = [\bar{A}(t) + \bar{B}(t)F(t)\bar{C}(t)]\bar{x}(t) + \bar{w}(t) + \bar{B}(t)F(t)\bar{v}(t) \tag{3-24}$$

where

$$\bar{x}(t) = \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}, \quad \bar{w}(t) = \begin{bmatrix} w(t) \\ 0_{p,1} \end{bmatrix}, \quad \bar{v}(t) = \begin{bmatrix} v_1(t) \\ 0_{r_2+p,1} \end{bmatrix},$$

$$\bar{A}(t) = \begin{bmatrix} A(t) & 0_{n,p} \\ 0_{p,n} & 0_{p,p} \end{bmatrix}, \quad \bar{B}(t) = \begin{bmatrix} B(t) & 0_{n,p} \\ 0_{p,m} & I_p \end{bmatrix},$$

$$\bar{C}(t) = \begin{bmatrix} C_1(t) & 0_{r_1,p} \\ C_2(t) & 0_{r_2,p} \\ 0_{p,n} & I_p \end{bmatrix}, \quad F(t) = \begin{bmatrix} G_1(t) & G_2(t) & H(t) \\ N_1(t) & N_2(t) & P(t) \end{bmatrix}.$$

The performance index (3-19) can then be expressed as

$$J = E\{\bar{x}'(t_f)\bar{S}\bar{x}(t_f) + \int_{t_o}^{t_f} \bar{x}'(t)[\bar{Q}(t) + \bar{C}'(t)F'(t)\bar{R}(t)F(t)\bar{C}(t)]\bar{x}(t)dt\} \tag{3-25}$$

where

$$\bar{S} \triangleq \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{Q}(t) \triangleq \begin{bmatrix} Q(t) & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{R}(t) \triangleq \begin{bmatrix} R(t) & 0 \\ 0 & 0 \end{bmatrix}.$$

(3-25) may be rewritten

$$J = \mathrm{tr}\{\bar{S}\bar{X}(t_f) + \int_{t_o}^{t_f} [\bar{Q}(t) + \bar{C}'(t)F'(t)\bar{R}(t)F(t)\bar{C}(t)]\bar{X}(t)dt\} \tag{3-26}$$

where

$$\bar{X}(t) = E\{\bar{x}(t)\bar{x}'(t)\}.$$

In view of (3-24), it can be readily shown [19] that the $(n+p)\times(n+p)$ matrix $\bar{X}(t)$ satisfies the differential equation

$$\begin{aligned}\dot{\bar{X}}(t) = {} & [\bar{A}(t) + \bar{B}(t)F(t)\bar{C}(t)]\bar{X}(t) + \bar{X}(t)[\bar{A}(t) + \bar{B}(t)F(t)\bar{C}(t)]' \\ & + \bar{W}(t) + \bar{B}(t)F(t)\bar{V}(t)F'(t)\bar{B}'(t)\end{aligned} \tag{3-27}$$

where

$$\bar{W}(t) = \begin{bmatrix} W(t) & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{V}(t) = \begin{bmatrix} V_1(t) & 0 \\ 0 & 0 \end{bmatrix},$$

with the boundary condition

$$\bar{X}(t_o) = E\left\{\begin{bmatrix} x(t_o) \\ z(t_o) \end{bmatrix}\begin{bmatrix} x(t_o) \\ z(t_o) \end{bmatrix}'\right\}. \tag{3-28}$$

Using (3-17) and (3-22), (3-28) may be evaluated to give

$$\bar{X}(t_o) = \begin{bmatrix} X_o & \mu_o\gamma_o' + X_oC_2'(t_o)\beta' \\ \gamma_o\mu_o' + \beta C_2(t_o)X_o & \gamma_o\gamma_o' + \gamma_o\mu_o'C_2'(t_o)\beta' + \beta C_2(t_o)\mu_o\gamma_o' \\ & + \beta C_2(t_o)X_oC_2'(t_o)\beta' \end{bmatrix} \tag{3-29}$$

The problem now is to find the matrix function $F(t)$ and constants γ_o, β that minimize the performance index (3-26) subject to (3-27) and (3-29). This is evidently a variational problem in matrix form with $\overline{X}(t)$ as the state matrix and $F(t)$ as the control matrix; the initial state $\overline{X}(t_o)$ is partially constrained by (3-29) while the final state $\overline{X}(t_f)$ is completely free.

The hamiltonian for this problem is

$$H(\overline{X},L,F,t) = \mathrm{tr}[\{\overline{Q}(t) + \overline{C}'(t)F'(t)\overline{R}(t)F(t)\overline{C}(t)\}\overline{X}(t) + \dot{\overline{X}}(t)L'(t)]$$

where $L(t)$ is the costate matrix and satisfies the differential equation

$$-\dot{L}(t) = \frac{\partial H}{\partial \overline{X}} = [\overline{A}(t) + \overline{B}(t)F(t)\overline{C}(t)]'L(t)$$

$$+ L(t)[\overline{A}(t) + \overline{B}(t)F(t)\overline{C}(t)] + \overline{Q}(t) + \overline{C}'(t)F'(t)\overline{R}(t)F(t)\overline{C}(t) \tag{3-30}$$

with the boundary condition

$$L(t_f) = \frac{\partial}{\partial \overline{X}} \mathrm{tr}(\overline{S}\overline{X})\bigg|_{t=t_f} = \overline{S}' = \overline{S} \tag{3-31}$$

3.3.3 Discussion on problem singularity

As pointed out by Mendel and Feather [3], the hamiltonian H is linear in the submatrix $P(t)$ of $F(t)$, and hence the variational problem in hand is singular in $P(t)$. Since there is no a priori bound on $P(t)$, the optimal $P(t)$ would, in general, consist of an alternating sequence of impulses and singular subarcs. From (3-16) to (3-18) it is seen that an impulse in $P(t)$ would cause a step change in

$z(t)$ and hence in the state matrix $\overline{X}(t)$ but with no immediate change in $x(t)$. An impulse at the initial time t_o would serve to transfer $\overline{X}(t)$ from its initial value to the first singular subarc, subsequent impulses would transfer $\overline{X}(t)$ from one singular subarc to another while an impulse at time t_f would transfer $\overline{X}(t)$ to its optimal final value.

It is now conjectured that the optimal $P(t)$ in fact contains no impulses and therefore comprises just one singular subarc. The arguments in favour of this conjecture are as follows: If $z(t_o)$ had been fixed, then it is very likely that $P(t)$ would contain an impulse at time t_o to transfer $z(t_o)$ to the optimal singular subarc; however, since $z(t_o)$ is free to be chosen optimally, the need for an impulse at time t_o is eliminated. Similarly, since $z(t_f)$ is completely free, there is no need for an impulse at time t_f. Finally, there seems no reason for the optimal $P(t)$ to jump from one singular subarc to another, and hence for the existence of impulses in $P(t)$ at intermediate times.

In any event, even if this conjecture were false and the optimal $P(t)$ did contain impulses, there would be no real difficulty from a practical standpoint. This is because any computational algorithm would, in effect, determine a piecewise-constant approximation of the optimal $P(t)$ that is necessarily finite because a $P(t)$ that is infinite over a non-vanishing time interval cannot possibly be optimal. Moreover, the deviation of the cost J from its truly optimal value may be made arbitrarily small by making the number of time steps in the piecewise-constant approximation sufficiently large.

The presence of singular subarcs in P(t) would appear to rule out the indirect approach [10] as a means for finding the optimal P(t), at least in its published form. However, as the results of Jacobsen and Lele [16] show, singular optimal controls may be efficiently computed using direct methods such as the conjugate gradient algorithm of Lasdon *et al.* [17] . The necessary gradients for use with this algorithm will therefore be evaluated next.

3.3.4 Derivation of expressions for gradients

From (3-26) to (3-31), the first variation δJ is clearly given by

$$\delta J = \mathrm{tr}[\delta\bar{X}(t_o)L'(t_o) + \int_{t_o}^{t_f} \frac{\partial H}{\partial F}\,\delta F'(t)dt] \qquad (3\text{-}32)$$

It is convenient to partition the symmetric matrix L as follows:

$$L = \begin{bmatrix} L_{xx} & L_{zx}' \\ L_{zx} & L_{zz} \end{bmatrix}$$

where the submatrices L_{xx}, L_{zx} and L_{zz} are of dimensions n×n, p×n and p×p respectively. Now using (3-29), (3-32) can be expanded to give

$$\delta J = \mathrm{tr}[2\{\mu_o\delta\gamma_o' + X_oC_2'(t_o)\delta\beta'\}L_{zx}(t_o) + 2\{\gamma_o\delta\gamma_o' + \beta C_2(t_o)\mu_o\delta\gamma_o'$$

$$+ \gamma_o\mu_o'C_2'(t_o)\delta\beta' + \beta C_2(t_o)X_oC_2'(t_o)\delta\beta'\}L_{zz}(t_o) + \int_{t_o}^{t_f} \frac{\partial H}{\partial F}\delta F'(t)dt$$

(3-33)

From (3-33), the required gradients are

$$\frac{\partial J}{\partial \gamma_o} = 2L_{zx}(t_o)\mu_o + 2L_{zz}(t_o)\{\gamma_o + \beta C_2(t_o)\mu_o\} \tag{3-34}$$

$$\frac{\partial J}{\partial \beta} = 2L_{zx}(t_o)X_oC_2'(t_o) + 2L_{zz}(t_o)\{\gamma_o\mu_o'C_2'(t_o) + \beta C_2(t_o)X_oC_2'(t_o)\} \tag{3-35}$$

Also,

$$\frac{\partial H}{\partial F} = 2\overline{R}(t)F(t)\overline{C}(t)\overline{X}(t)\overline{C}'(t) + 2\overline{B}'(t)L(t)\overline{X}(t)\overline{C}'(t) + 2\overline{B}'(t)L(t)\overline{B}(t)F(t)\overline{V}(t) \tag{3-36}$$

3.3.5 Solution algorithm

A suitable computational scheme for finding the solution would therefore consist of the following steps.

Step (i): Choose initial estimates for γ_o, β and $F(t)$, ensuring that the submatrix $G_1(t)$ of $F(t)$ is a null matrix.

Step (ii): Evaluate $\overline{X}(t_o)$ using (3-29). Then integrate the linear matrix equation (3-27) forwards in time from $t = t_o$ to $t = t_f$.

Step (iii): Integrate the linear matrix equation (3-30) backwards in time from $t = t_f$ to $t = t_o$ with the boundary condition given by (3-31).

Step (iv): Evaluate the quantities $\frac{\partial J}{\partial \gamma_o}$, $\frac{\partial J}{\partial \beta}$ and $\frac{\partial H}{\partial F}$ using (3-34) to (3-36).

Step (v): Update the estimates of γ_o, β and $F(t)$ in accordance with the conjugate gradient algorithm. The submatrix $G_1(t)$ of $F(t)$ is not treated as a variable for this purpose.

Step (vi): Test for convergence by comparing the decrease in cost over the iteration against a prescribed tolerance. Return to step (ii) if the test fails.

The class of problems considered in this chapter is known (see e.g. [8]) to exhibit local optima. Hence, it is advisable to repeat the computation using widely differing initial estimates of γ_o, β and $F(t)$ in order to increase the likelihood of finding the global optimum.

3.3.6 Suboptimal compensators

For systems of even moderate complexity, the implementation of the time-varying compensator (3-18) would present formidable practical difficulties. This is because a total of $p(p+r_1+r_2) + m(p+r_2)$ functions of time have to be stored and then "played back" synchronously. Hence it is desirable to seek suboptimal solutions that are simpler to implement. One such solution may be obtained by applying the Ritz method of approximation, as Kleinman and Athans [12] have done for the complete state feedback problem.

3.3.6.1 General formulation

In this approach, the compensator gains are constrained to be of the form

$$F(t) = \sum_{k=1}^{M} \phi_k(t)\, F_k \tag{3-37}$$

where the $\phi_k(t)$, $k = 1,2,\ldots,M$ are an arbitrary preselected set of linearly independent scalar time functions and the F_k, $k = 1,2,\ldots,M$ are constant matrices. The original dynamic optimization problem in the function $F(t)$ (as well as γ_o and β) then reduces to a static optimization problem in the variables γ_o, β and F_k, $k = 1,2,\ldots,M$. This latter problem may be solved numerically using standard gradient techniques, such as the algorithm of Davidon [18], providing the gradient of the performance index (3-19) with respect to these variables can be computed.

Expressions have already been obtained for $\frac{\partial J}{\partial \gamma_o}$ and $\frac{\partial J}{\partial \beta}$. To determine $\frac{\partial J}{\partial F_k}$, substitute from (3-37) in (3-32) to obtain

$$\delta J = \mathrm{tr}\left[\delta\bar{X}(t_o)L'(t_o) + \sum_{k=1}^{M}\int_{t_o}^{t_f}\frac{\partial H}{\partial F}\,\phi_k(t)\,\delta F_k'\,dt\right],$$

whence

$$\frac{\partial J}{\partial F_k} = \int_{t_o}^{t_f}\frac{\partial H}{\partial F}\,\phi_k(t)\,dt \tag{3-38}$$

where $\frac{\partial H}{\partial F}$ is given by (3-36).

The number M of basis functions required to give acceptable suboptimal performance depends on the actual problem and the set of functions $\{\phi_k(t)\}$ chosen. If the set of basis functions is complete, the suboptimal performance may be made as close as desired to the optimal performance by making M sufficiently large.

3.3.6.2 Piecewise-constant gains

One practically important choice of basis functions is that employed by Kleinman, Fortmann and Athans [13]. The time interval (t_o,t_f) is arbitrarily partitioned as follows:

$$t_o < t_1 < t_2 \ldots < t_M \triangleq t_f ,$$

and the function $\phi_k(t)$ is defined by

$$\phi_k(t) = \begin{cases} 1, & t_{k-1} \leqslant t \leqslant t_k \\ 0, & \text{otherwise} \end{cases} \tag{3-39}$$

With this choice, the compensator gains become piecewise-constant functions of time with step changes at the times t_k, $k=1,2,\ldots,M-1$, i.e.,

$$F(t) = F_k, \quad t_{k-1} \leqslant t \leqslant t_k, \quad k=1,2,\ldots,M. \tag{3-40}$$

The expression for the gradient (3-38) now simplifies to

$$\frac{\partial J}{\partial F_k} = \int_{t_{k-1}}^{t_k} \frac{\partial H}{\partial F}\, dt \tag{3-41}$$

Kleinman *et al.* [13] mention the desirability of choosing the set of times $\{t_k\}$ so as to minimize J, instead of specifying it arbitrarily at the outset. Although they do not give a method for doing so, this may be done easily as follows. The times t_k, $k=1,2,\ldots,M-1$, now become additional variables for the derived static optimization problem which may still be solved using a gradient technique providing the set of gradients $\{\frac{\partial J}{\partial t_k}\}$ is available. To obtain these quantities, observe that when perturbation of $\{t_k\}$ is allowed for, the first variation δJ becomes

$$\delta J = \operatorname{tr}\Bigg[\delta\bar{X}(t_o)L'(t_o) + \sum_{k=1}^{M} \int_{t_{k-1}}^{t_k} \frac{\partial H}{\partial F}\, \delta F_k'\, dt$$

$$+ \sum_{k=1}^{M-1} \{H(\bar{X}(t_k),L(t_k),F_{k-1},t_k) - H(\bar{X}(t_k),L(t_k),F_k,t_k)\}\delta t_k\Bigg].$$

Hence,

$$\frac{\partial J}{\partial t_k} = H(\overline{X}(t_k),L(t_k),F_{k-1},t_k) - H(\overline{X}(t_k),L(t_k),F_k,t_k). \quad (3\text{-}42)$$

3.3.6.3 Algorithm for piecewise-constant gains

The following algorithm is now proposed for obtaining the piecewise-constant suboptimal compensator.

Step (i): Choose priming values for γ_o, β, F_k, $k=1,2,\ldots,M$, and t_k, $k=1,2,\ldots,M-1$, ensuring that the $m \times r_1$ submatrix of each F_k corresponding to $G_1(t)$ is a null matrix.

Step (ii): Evaluate $\overline{X}(t_o)$ using (3-29). Then integrate the matrix equation (3-27) forwards in time from $t = t_o$ to $t = t_f$.

Step (iii): Integrate the matrix equation (3-30) backwards in time from $t = t_f$ to $t = t_o$ with the boundary condition given by (3-31).

Step (iv): Evaluate the quantities $\frac{\partial J}{\partial \gamma_o}$, $\frac{\partial J}{\partial \beta}$, $\frac{\partial J}{\partial F_k}$, $k=1,2,\ldots,M$, and $\frac{\partial J}{\partial t_k}$, $k=1,2,\ldots,M-1$, using (3-34), (3-35), (3-41) and (3-42) respectively.

Step (v): Update the estimates of γ_o, β, F_k, $k=1,2,\ldots,M$, and t_k, $k=1,2,\ldots,M-1$, in accordance with, say, the Davidon-Fletcher-Powell algorithm. The submatrices of the F_k corresponding to $G_1(t)$ are not treated as variables for this purpose, and so remain null.

Step (vi): Apply an appropriate convergence test. Return to step (ii) if the test fails.

Again it is advisable to repeat the computation using widely differing initial iterates in order to increase the likelihood of locating the global optimum.

Note that it is not essential that the times t_k be made variable. If it is desired to fix them at the outset, all that need be done is to delete them from the set of variables. The same applies to the quantities γ_o and β. If, for instance, it is considered sufficient that the initial compensator state $z(t_o)$ be set to zero, then γ_o and β are both set to zero and deleted from the set of variables.

3.3.7 Numerical example

An illustrative numerical example will now be presented. The plant equations are

$$\dot{x}(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + w(t)$$

$$y(t) = [0 \quad 1] \; x(t) + v(t)$$

and the performance index is

$$J = E\{x'(2)Sx(2) + \int_0^2 [x'(t)Qx(t) + u'(t)Ru(t)]dt\}$$

where

$$E\{x(0)\} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}, \quad E\{x(0)x'(0)\} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$E\{w(t)\} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad E\{w(t)w'(\tau)\} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \delta(t-\tau)$$

$$E\{v(t)\} = 0, \quad E\{v(t)v'(\tau)\} = [1]\,\delta(t-\tau)$$

$$S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = [0.1]$$

Here $r_2 = 0$ since there is no noise-free observation. Hence there is no non-trivial solution for $p = 0$, i.e., memoryless output feedback. Non-trivial solutions are, however, possible for $p \geqslant 1$. The optimal (unrestricted-order) compensator is, of course, a Kalman filter followed by a memoryless linear transformation and has order $p = 2$.

The problem considered is the design of optimal piecewise-constant first-order compensators. Since $r_2 = 0$, the matrices $G_2(t)$, $N_2(t)$ and β clearly do not exist for this problem. The redundancy in the remaining compensator parameters discussed in §3.2.1 may be removed by arbitrarily fixing $N_1(t) = 1$.

The problem was solved using the algorithm of §3.3.3 for the cases $M = 1$, $M = 5$ and $M = 10$, where M is the number of (equal) subintervals into which the time interval $(0,2)$ is subdivided. The differential equations were integrated using a fourth-order Runge-Kutta routine while the definite integrals were evaluated using Simpson's rule. The minimizations were performed using the Davidon-Fletcher-Powell algorithm and in every case it was found that all iterates

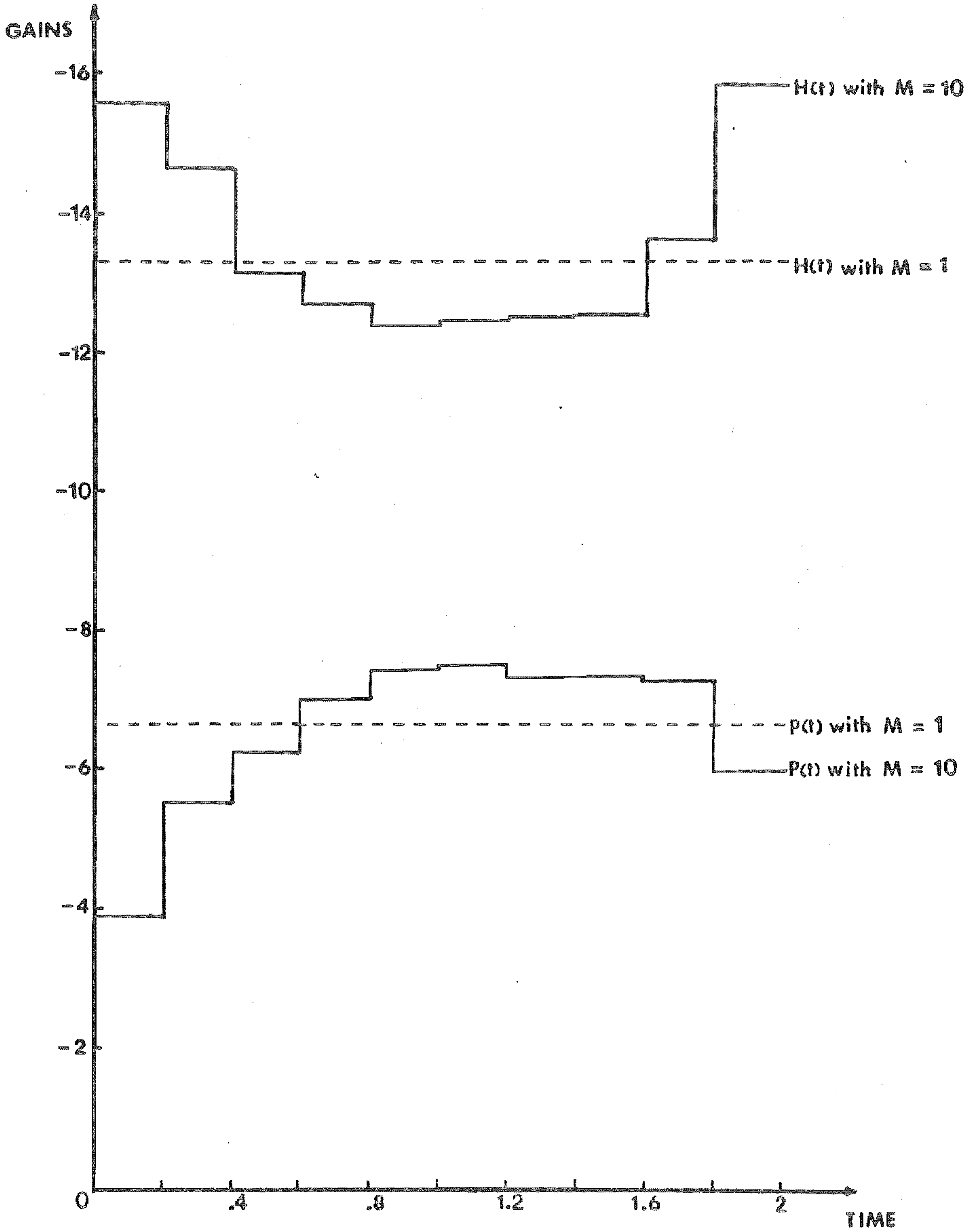

Fig. 3.2 Optimal profiles of compensator parameters P(t) and H(t)

converged to a unique solution. The results obtained are summarised in the table below that gives the minimum cost J and optimal $\gamma_o \equiv z(0)$ for the three cases, and Fig. 3.2 that shows the optimal profiles of the free compensator parameters P(t) and H(t) for the two cases M = 1 and M = 10. The results clearly indicate rapid convergence with increasing M, and the solution for M = 10 must therefore be very close to the strictly optimal time-varying solution.

M	J_{min}	$z(0)_{opt}$
1	2.2073	0.0249
5	2.1083	-0.0115
10	2.1041	-0.0107

3.4 CONCLUSION

A method has been presented for designing fixed-order, fixed-configuration time-invariant compensators for stationary stochastic LMS that are optimal wrt a quadratic performance index. Plant input noise and observation noise, neither of which necessarily be white, were both allowed for. Acceptable control of such systems may thereby be achieved without the need for high-dimensional state estimators and/or optimal controllers.

The fixed-configuration non-stationary stochastic linear quadratic optimal control problem has also been treated in some generality. The compensator was constrained to be a fixed-order but possibly time-varying dynamic system and a matrix variational problem in the compensator gains and parameters defining the initial compensator state was derived. Although this problem was singular in one of the compensator gains, it was shown that a solution could be obtained using standard conjugate-gradient algorithms and the necessary theory for this purpose was developed.

An algorithm was also given for obtaining suboptimal compensators whose gains are constrained to be piecewise-constant functions of time, with provision for optimally choosing the instants at which the gain changes occur.

The algorithms presented in this chapter for both stationary and non-stationary stochastic LMS are similar to that used in Chapter 2 for deterministic LMS. Being based on the conjugate gradient and Davidon-Fletcher-Powell algorithms, they may be expected to exhibit good convergence properties. This is borne out by the computational results obtained.

REFERENCES

[1] Joseph, P.D. and Tou, J.T., "On linear control theory," AIEE Trans. (Appl. Ind.), vol. 80, pp.193-196, Sept. 1961.

[2] Sirisena, H.R. and Choi, S.S., "Design of optimal constrained dynamic compensators for linear stationary stochastic servomechanisms," Int. J. Contr., vol. 20, pp.363-368, Sept. 1974.

[3] Mendel, J.M. and Feather, J., "On the design of optimal time-invariant compensators for linear stochastic time-invariant systems," IEEE Trans. Automat. Contr., vol. AC-20, pp.653-657, Oct. 1975.

[4] Basuthakur, S. and Knapp, C.H., "Optimal constant controllers for stochastic linear systems," IEEE Trans. Automat. Contr., vol. Ac-20, pp.664-666, Oct. 1975.

[5] Kurtaran, B.Z., "Suboptimal control for discrete linear constant stochastic systems," IEEE Trans. Automat. Contr., vol. AC-20, pp.423-425, June 1975.

[6] Newton, G.C., Jr., Gould, L.A. and Kaiser, J.F., Analytical Design of Linear Feedback Controls. New York: Wiley, 1957.

[7] Åström, K.J., Introduction to Stochastic Control Theory. New York: Academic Press, 1970.

[8] Axsäter, S., "Sub-optimal time-variable feedback control of linear dynamic systems with random inputs," Int. J. Contr., vol. 4, pp.549-566, Dec. 1966.

[9] Sidar, M. and Kurtaran, B.Z., "Optimal low-order controllers for linear stochastic systems," Int. J. Contr., vol. 3, pp.377-387, Sept. 1975.

[10] Sims, C.S. and Melsa, J.L., "A fixed configuration approach to the stochastic linear regulator problem," in Proc. Joint Automatic Control Conf., Atlanta, Ga., pp.708-712, June 22-26, 1970.

[11] Bryson, A.E. Jr. and Johansen, D.E., "Linear filtering for time-varying systems using measurements containing coloured noise," IEEE Trans. Automat. Contr., vol. AC-10, pp.4-10, Jan. 1965.

[12] Kleinman, D.L. and Athans, M., "The design of suboptimal linear time-varying systems," IEEE Trans. automat. Contr., vol. AC-13, pp.150-159, Apr. 1968.

[13] Kleinman, D.L., Fortmann, T. and Athans, M., "On the design of linear systems with piecewise-constant feedback gains," IEEE Trans. Automat. Contr., vol. AC-13, pp.354-361, Aug. 1968.

[14] Sirisena, H.R. and Choi, S.S., "Design of optimal constrained dynamic compensators for nonstationary linear stochastic systems," to appear in Int. J. Contr.

[15] Johnson, T.L. and Athans, M., "On the design of optimal constrained dynamic compensators for linear constant systems," IEEE Trans. Automat. Contr., vol. Ac-15, pp.658-660, Dec. 1970.

[16] Jacobson, D.H. and Lele, M.M., "A transformation technique for optimal control problems with a state variable inequality constraint," IEEE Trans. Automat. Contr., vol. AC-14, pp.457-464, Oct. 1969.

[17] Lasdon, L.S. Mitter, S.K. and Waren, A.D., "The conjugate gradient method of optimal control problems," IEEE Trans. Automat. Contr., vol. Ac-12, pp.132-138, April 1967.

[18] Fletcher, R. and Powell, M.J.D., "A rapidly convergent descent method for minimization," British Computer J., pp.163-168, June 1963.

[19] Athans, M. and Tse, E., "A direct derivation of the optimal linear filter using the maximum principle," IEEE Trans. Automat. Contr., vol. AC-12, pp.690-698, Dec. 1967.

CHAPTER 4

POLE PLACEMENT IN LINEAR MULTIVARIABLE SYSTEMS USING DYNAMIC OUTPUT FEEDBACK

4.1 INTRODUCTION

In the previous two chapters, attention has been directed towards the design of output feedback controllers for both noise-free and noisy regulator systems via the minimization of quadratic performance indices in the state and control variables. However, as has been mentioned in Chapter 1, alternative design approaches exist, one that has received much attention being the *pole placement* (or *modal control*) approach discussed in §1.2.4.

Initial research efforts in this area were concentrated on the use of complete state feedback laws, see e.g. [1]. An important contribution was made by Wonham [2] who showed that if the LMS

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}\right\} \qquad (4\text{-}1)$$

is controllable and observable, then there always exists a matrix F for the feedback law

$$u(t) = Fx(t) + u_e(t) \qquad (4\text{-}2)$$

where $u_e(t)$ is an external input, such that the poles of the closed-loop system

$$\dot{x}(t) = (A + BF)x(t) + Bu_e(t) \tag{4-3}$$

assume arbitrary values subject to complex conjugate pairing.

Pole placement using *output* feedback is, however, a more practical and interesting problem. Researchers working in this area are usually confronted with the following questions: for a given plant of order n and compensator of order p, see Fig. 4.1,

(a) What is the minimum p such that all n + p closed-loop poles can have any pre-assigned values?

(b) What is the most efficient way of obtaining the corresponding minimal-order compensator?

In fact, it cannot be claimed that the problem has been completely solved since at this writing, no explicit expression on the minimum order p discussed in (a) has been obtained. For the LMS (4-1), Davison and Chatterjee [3] have shown that max (m,r) closed-loop poles may be arbitrarily assigned using constant output feedback (i.e., p = 0), where $m = \rho[B]$ and $r = \rho[C]$. An algorithm to construct such a gain matrix can be found in [4]. An extension of the results in [3] appears in [5]; for "almost all" (B,C) pairs it has been established that min (n,m + r-1) poles can be assigned arbitrarily close to prescribed values using constant output feedback. This result has also been obtained independently in [6].

Even greater control over the closed-loop pole locations may be achieved using feedback through a dynamic compensator. Thus Ahmari and Vacroux [7] have generalized the result in [3] to show that

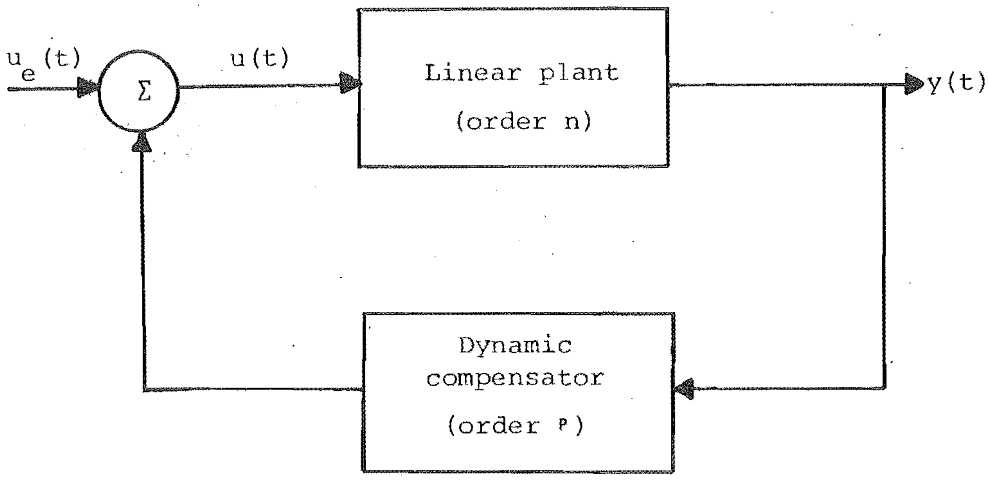

Fig. 4.1 Linear multivariable control system

$$q = \max(\alpha + p, \beta + p) \tag{B1}$$

closed-loop poles may be arbitrarily assigned using a p-th order dynamic compensator, where

$$\alpha = \rho[C', A'C', \ldots, (A')^{p}C']$$

$$\beta = \rho[B, AB, \ldots, A^{p}B]$$

It also follows from (B1) that provided the plant (4-1) is both controllable and observable, all $n + p$ poles of the closed-loop system may be arbitrarily assigned using a compensator of order

$$p = \min(\nu_o - 1, \nu_c - 1) \tag{B2}$$

where ν_o, ν_c are respectively the observability and controllability indices of the plant (4-1). This latter result was first obtained in [8]. An alternative upper bound on p has also been determined in [6] and it is given by

$$p = n - m - r + 1 \tag{B3}$$

This bound is easier to compute than that shown in (B2). Combining (B2) and (B3), therefore an upper bound on p for exact pole placement of all closed-loop poles is

$$p = \min[\nu_o - 1, \nu_c - 1, n - m - r + 1] \tag{B4}$$

Actually, it may be shown that the dynamic compensator structures considered in [6], [7] and [8] are somewhat restrictive and therefore have fewer degrees of design freedom than the maximum possible for a given order of compensator. Hence, (B1) only represents a lower bound on the number of poles that may be assigned arbitrarily.

Similarly, (B4) is only an upper bound on the compensator order required for complete pole placement. Thus, an example where all the closed-loop poles may be assigned using a compensator of order less than that given in (B4) is presented in §4.2.5 of this chapter.

References [9] - [12] among others also propose pole placement algorithms which require the imposition of unity-rank restrictions on some or all of the compensator matrices; again the consequent loss of design freedom generally means that non-minimal order compensators are obtained.

In this chapter, a new least-squares design technique for pole placement using a dynamic compensator is presented. Some of these results were first described in Sirisena and Choi [13] in a slightly different form. Unlike all the aforementioned references, there are no restrictions on the compensator structure except that it be linear and time-invariant. The pole placement problem is formulated in §4.2.1. This is followed in §4.2.2 by a derivation of a lower bound on the minimal compensator order needed for exact pole placement. Analytical expressions required for the solution of the problem using function minimization techniques are developed in §4.2.3. These expressions are then used in conjunction with the lower bound obtained in §4.2.2 in the new computational algorithm described in §4.2.4 that enables exact pole placement to be achieved with a compensator of the lowest possible order. Numerical results obtained with the new algorithm are presented in §4.2.5.

An alternative pole-placement algorithm for the case of unrestricted-rank constant output feedback has been independently derived by Patel [14]. He treats the multi-input system as a series of single-input systems and this leads to an iterative algorithm in which the rows of the feedback matrix are updated one at a time. It would appear therefore that his algorithm may be less efficient computationally than the algorithm proposed here which simultaneously updates all the rows of the feedback matrix.

Consideration is given in §4.3.1 to the more general problem of achieving either exact or approximate pole placement while minimizing a quadratic performance index in the control and state variables. The corresponding problem for complete state feedback has been treated in [15]. Reference [16] have also considered a similar problem using output feedback; however, their compensator structure lacks generality and their performance index weights certain compensator parameters rather than the actual plant inputs.

The requirement of exact pole placement is relaxed in §4.4 where the poles are required to be positioned within a prescribed region of the complex plane only. The results obtained in this section first appeared in a paper by Sirisena and Choi [17].

4.2 POLE PLACEMENT PROBLEM

The plant under consideration is described by equation (4-1), where $x(t) \varepsilon R^n$, $y(t) \varepsilon R^r$ and $u(t) \varepsilon R^m$ are the plant state, output and control vectors respectively. A, B, C are constant matrices and there is no loss of generality in assuming that both B and C have full rank.

The problem is to design a pth-order dynamic compensator of the form

$$\left.\begin{aligned} \dot{z}(t) &= Pz(t) + Ny(t) \\ u(t) &= Hz(t) + Gy(t) \end{aligned}\right\} \qquad (4\text{-}4)$$

where z(t) is the compensator state and P, N, H and G are appropriately dimensioned constant matrices (see Fig. 4.2).

In view of (4-1) and (4-6), the closed-loop system may be described by the composite equation

$$\dot{\bar{x}}(t) = (\bar{A} + \bar{B}F\bar{C})\bar{x}(t) \triangleq \bar{A}_o\bar{x}(t) \qquad (4\text{-}5)$$

where

$$\left.\begin{aligned} &\bar{x}(t) \triangleq \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}; \qquad \bar{A} = \begin{bmatrix} A & 0_{n,p} \\ 0_{p,n} & 0_{p,p} \end{bmatrix}; \\ &\bar{B} = \begin{bmatrix} B & 0_{n,p} \\ 0_{m,p} & I_p \end{bmatrix}; \quad \bar{C} \triangleq \begin{bmatrix} C & 0_{r,p} \\ 0_{n,p} & I_p \end{bmatrix}; \quad F = \begin{bmatrix} G & H \\ N & P \end{bmatrix} \end{aligned}\right\} \qquad (4\text{-}6)$$

4.2.1 Problem formulation

The closed-loop poles λ_i, $i = 1,\ldots,n+p$ are the roots of the characteristic equation

$$\Delta(\lambda) \triangleq |\lambda I_{n+p} - \bar{A}_o| \equiv 0 \qquad (4\text{-}7)$$

Suppose it is desired to make q, $0 \leqslant q \leqslant n+p$, poles of the closed-loop system (4-5) either exactly or approximately equal to q predetermined values λ_i^d, $i = 1,\ldots,q$. Suppose $k(\leqslant q)$ of these desired poles are real while the remainder are complex and appear in conjugate pairs. Let the poles be ordered as follows

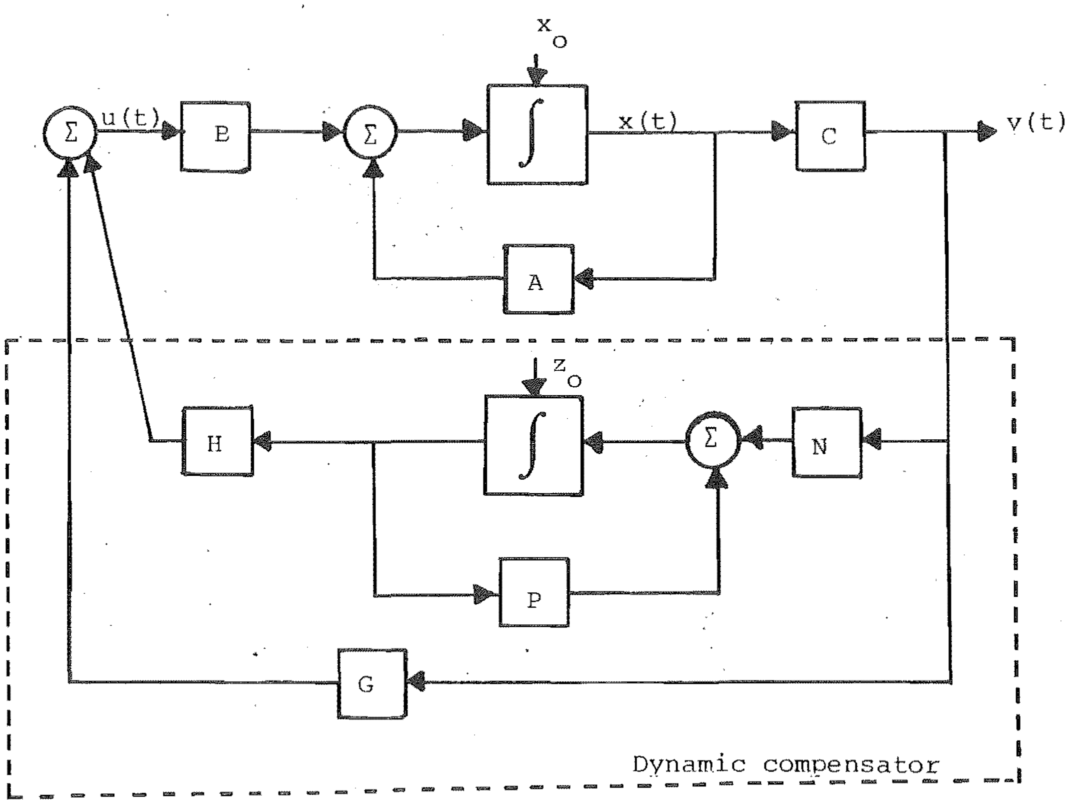

Fig. 4.2 The closed-loop linear multivariable system

$$\mathrm{Im}(\lambda_i^d) = 0, \qquad i = 1,\ldots,k$$

$$\mathrm{Im}(\lambda_i^d) > 0, \qquad i = k+1,\ldots,l$$

$$\mathrm{Im}(\lambda_i^d) > 0, \qquad i = l+1,\ldots,q$$

where of course k, l and q are related by

$$l = (q + k)/2$$

Subsequently, it is assumed that the λ_i^d are all distinct. The case where some of the λ_i^d are equal is discussed in Appendix A.

The design constraint on the compensator matrix F may now be stated as

$$\lambda_i = \lambda_i^d , \qquad i = 1,\ldots,q \tag{4-8}$$

Constraint (4-8) may be expressed in several equivalent forms. Firstly (4-8) implies and is implied by

$$\Delta(\lambda) = \Delta_d(\lambda) \triangleq \prod_{i=1}^{q} (\lambda - \lambda_i^d) \tag{4-9}$$

where $\Delta_d(\lambda)$ is termed the *desired closed-loop characteristic polynomial.*

Moreover, (4-9) implies that

$$\Delta(\lambda_i^d) = 0 \; , \qquad i = 1,\ldots,q \tag{4-10}$$

It is convenient to replace (4-10) by the single constraint

$$\sum_{i=1}^{q} |\Delta(\lambda_i^d)|^2 = 0 \tag{4-11}$$

The LHS of (4-11) will be recognised as the least-squares cost function used by Ahmari and Vacroux [7] and Sirisena and Choi [13]. Actually, since

$$|\Delta(\lambda_i^{d*})| = |\Delta(\lambda_i^d)|$$

where the asterisk denotes the complex conjugate, define a performance index

$$J_1(F) = \sum_{i=1}^{l} S_i^{\,2} |\Delta(\lambda_i^d)|^2 \tag{4-12}$$

where the summation is over only l desired poles that satisfy

$$\mathrm{Im}(\lambda_i^d) \geqslant 0$$

and S_i, $i = 1,\ldots,l$ are real constants.

The pole placement problem may now be stated.

Statement of pole placement problem (S1)

Given the system (4-1) and the pth-order compensator (4-4), and a set of desired poles λ_i^d, $i = 1,\ldots,q$, find the parameter matrix F such that $J_1(F)$ is minimized.

It is clear that the q closed-loop poles will be exactly placed on the desired locations if and only if $J_1 = 0$.

Remark

(R1). Alternatively when $q = n + p$, the pole placement cost $\hat{J}_1$ may be expressed as the sum of squares of the differences between the coefficients in $\Delta(\lambda)$, as defined in (4-7), and the corresponding coefficients in $\Delta_d(\lambda)$, as defined in (4-9). Clearly, $\hat{J}_1 = 0$ is satisfied if and only if $\lambda_i = \lambda_i^d$, $i = 1,\ldots,n + p$. Computable expressions for $\hat{J}_1$ and $\partial\hat{J}_1/\partial F$ required in the solution algorithm (see §4.2.2) can be readily obtained using the recursive relationship (4-21) and (4-22) shown in §4.2.3.1. However, this new approach will not be considered here because

(i) the expression $\partial\hat{J}_1/\partial F$ is more complicated than that obtained in §4.2.3 and therefore may be computationally unattractive and

(ii) with $\hat{J}_1$ as defined, it is only meaningful to consider the special case $q = n + p$. This loss of design generality is especially undesirable in cases where it is only necessary to shift those (q) poles which are dominant.

Problem (S1) stated above is a parameter optimization problem, the solution of which may be obtained using standard minimization techniques. For reasons of speed and accuracy of solution, it is preferable to employ an algorithm which uses gradients. The necessary expressions will be derived later in §4.2.3. In the next section a lower bound on compensator order for exact pole placement is derived.

4.2.2 Minimum compensator order for exact pole placement

4.2.2.1 Number of independent parameters

Following the same arguments used in §2.2.2, it is seen that there exist p^2 redundant parameters in F and hence, the number of independent parameters contained in F is only $pm + mr + pr$. The designer may therefore adopt the canonical form (C1) and (C2) defined in Chapter 2 for the submatrices P and N so that the dimensionality of the parameter optimization problem is reduced to the bare minimum.

Notice that in [7] - [12], additional unity-rank restrictions are imposed on the compensator structure, either explicitly or implicitly. Although these restrictions may facilitate the implementation of the compensator, they are nevertheless accompanied by a considerable loss of design freedom. For example in [7], the multi-input plant (4-1) has been (in effect) transformed into a single-input plant by constraining all the inputs to be constant multiples of each other. Thus when the compensator is expressed in the state variable form (4-4), it can be shown that the matrices H,G have the unity-rank forms

$$H = ch', \qquad G = cg'$$

where c is a predetermined constant vector. Hence, the number of independent parameters in the compensator has been reduced to only $pr + p + r$ or $pm + p + m$ depending on whether the plant (4-1) or its dual is being compensated.

4.2.2.2 A lower bound on compensator order

No explicit expression is as yet available for the minimum order of compensator required for exact pole placement of a given number q, $0 \leqslant q \leqslant n + p$, of the closed-loop poles. However, from the discussion above, a lower bound on the compensator order will now be derived.

Exact pole placement of q closed-loop poles implies that the q relationships (4-10)

$$\Delta(\lambda_i^d) = 0, \qquad i = 1,\ldots,q$$

are satisfied. This is possible for arbitrary λ_i^d only if there are *at least* q effective degrees of design freedom. Now it has been shown (in §4.2.2.1) that a pth order compensator has a maximum of $pm + pr + mr$ independent parameters. However, for particular values of the plant matrices A, B and C, some of these parameters may not affect the system poles. In such cases, there would be fewer effective degrees of design freedom. Hence the only definite statement that can be made is that

$$q_{max} = pm + pr + mr \tag{4-13}$$

is an upper bound on the number of system poles that may be arbitrarily assigned using a pth order compensator. A corollary to this result is that

$$p \geqslant \frac{q - mr}{m + r} \tag{4-14}$$

is a *lower bound* on the compensator order for exact placement of q system poles. Also, from (4-13) it follows that a lower bound on the compensator order for exact placement of all $n+p$ system poles is

$$p \geqslant \frac{n - mr}{m + r - 1} \tag{4-15}$$

The lower bounds developed here will be used in the computational procedure to be described later in §4.2.4.

4.2.3 Solution technique

Attention is now focussed on the solution of the pole placement problem (S1). Observe that the cost J_1 may be expressed in the form

$$J_1 = h'h \tag{4-16}$$

where h is a real q-vector that is defined as

$$\begin{aligned} h \triangleq [&S_1\Delta(\lambda_1^d), S_2\Delta(\lambda_2^d), \ldots, S_k\Delta(\lambda_k^d), S_{k+1}\mathrm{Re}\{\Delta(\lambda_{k+1}^d)\}, \\ &S_{k+1}\mathrm{Im}\{\Delta(\lambda_{k+1}^d)\}, \ldots, S_l\mathrm{Re}\{\Delta(\lambda_l^d)\}, S_l\mathrm{Im}\{\Delta(\lambda_l^d)\}] \end{aligned} \tag{4-17}$$

Now if F is stored as an $(m+p)(p+r) \times 1$ column vector, then the $(m+p)(p+r) \times q$ matrix N is defined by

$$N = \frac{\partial}{\partial F} h' \tag{4-18}$$

Either of the following gradient-type algorithm may be used:

(1) The sum of square function (4-16) can be minimized with Gauss-Newton iteration technique [19]. The increment δF is given by the following formulae

$$\delta F = \begin{cases} -(NN')^{-1}Nh & \text{if } (m+p)(p+r) \leqslant q \\ -N(N'N)^{-1}h & \text{if } (m+p)(p+r) > q \end{cases} \tag{4-19}$$

In practice, better convergence is usually obtained using the modified version of formulae (4-19) proposed by Marquardt [20], see Appendix B.

(2) Alternatively, δF may be determined with the Davidon-Fletcher-Powell (DFP) algorithm [18]. For this purpose, an expression is required for $\partial J_1/\partial F$ and this is given by

$$\frac{\partial J_1}{\partial F} = 2\,Nh \tag{4-20}$$

It now remains to derive procedures for computing h and N. This is done in the next two sections.

4.2.3.1 Computation of the vector h

To obtain the vector h, it is necessary to evaluate $\Delta(\lambda)$ at the current value of F. The first step is to compute the coefficients a_i of $\Delta(\lambda)$ where

$$|\lambda I_{n+p} - \bar{A}_o| = \Delta(\lambda) \triangleq \lambda^{n+p} + a_1\lambda^{n+p-1} + \ldots + a_{n+p} \tag{4-21}$$

This can be easily done using the algorithm of Faddeev [21]:

$$\left.\begin{aligned} a_i &= -\frac{1}{i}\,\mathrm{tr}\{\phi_{i-1}\,\bar{A}_o'\} \\ \phi_i &= \phi_{i-1}\,\bar{A}_o' + a_i I_{n+p} \end{aligned}\right\} \quad i=1,\ldots,n+p \tag{4-22}$$

where $\phi_o = I_{n+p}$. It is also possible to obtain the coefficients a_i using the method of Davilevsky [22].

The elements of h can now all be obtained by repeated substitution of $\lambda = \lambda_i^d$, $i=1,\ldots,l$ in (4-21).

4.2.3.2 Computation of the matrix N

The matrix N is composed of elements of the form $\partial\Delta(\lambda_i^d)/\partial F$ and expressions for these quantities will now be obtained.

Firstly, it can be readily shown that [21]

$$\frac{\partial\Delta(\lambda)}{\partial F} = -\bar{B}'\,\Phi(\lambda)\bar{C}' \tag{4-23}$$

where $\Phi(\lambda)$ is the matrix of cofactors of the determinant $|\lambda I_{n+p} - \bar{A}_o|$, i.e. the transpose of the adjoint matrix of $\bar{A}_o$. Now $\Phi(\lambda)$ is computable because it may be expressed as [21]

$$\Phi(\lambda) = I_{n+p}\lambda^{n+p-1} + \phi_1\lambda^{n+p-2} + \ldots + \phi_{n+p-1} \tag{4-24}$$

and the matrix coefficients $\phi_1, \phi_2, \ldots, \phi_{n+p-1}$ are generated by the Faddeev algorithm during computation of the coefficients of the characteristic polynomial a_i, i.e. equation (4-21).

Therefore, the matrix N can be constructed by evaluating the quantities (4-24) for $\lambda = \lambda_i^d$, $i=1,\ldots,l$ with the aid of equations (4-21). Notice that

$$\left.\begin{aligned}
\frac{\partial}{\partial F}\,\mathrm{Re}\{\Delta(\lambda)\}\Big|_{\lambda=\lambda_j^d} &\equiv \mathrm{Re}\{\frac{\partial\Delta(\lambda)}{\partial F}\}\Big|_{\lambda=\lambda_j^d} \\
\frac{\partial}{\partial F}\,\mathrm{Im}\{\Delta(\lambda)\}\Big|_{\lambda=\lambda_j^d} &\equiv \mathrm{Im}\{\frac{\partial\Delta(\lambda)}{\partial F}\}\Big|_{\lambda=\lambda_j^d}
\end{aligned}\right\}\text{for } j = k+1,\ldots,l$$

Sufficient conditions have now been obtained for use in the solution algorithm shown in the next section.

4.2.4 Algorithm to determine the minimal-order compensator

The minimal-order compensator for exact pole placement may be determined using the following computational procedure.

Step (i): Choose the compensator order to be the smallest non-negative integer satisfying the appropriate inequality (4-14) or (4-15).

Step (ii): Guess an arbitrary starting value for F.

Step (iii): Minimize J_1 using the results of §4.2.3 in conjunction with a gradient-type optimization method, e.g. the DFP or Marquardt algorithm. If the minimum value of J_1 tends to zero, then exact pole placement has been attained and p is the required minimal order. Otherwise go to step (iv).

Step (iv): Either (a) choose a vastly different starting value for F and return to step (iii) or (b) increase p by one and go to step (ii). This course is followed only after repeated failures with course (a).

In view of the discussion in §4.1, for an observable and controllable plant (4-1), this procedure will definitely terminate at a finite p satisfying the inequality

$$p \leqslant \min\,(\nu_o - 1,\ \nu_c - 1,\ n - m - r + 1) \tag{4-25}$$

where the RHS of (4-25) is the upper bound given in (B4).

Finally, it is well known [25] that there may exist certain pole combinations that cannot be attained with finite compensator gains. However, in such cases, there always exists a value of p satisfying (4-25) for which the error measure J_1 may be made arbitrarily small, though not exactly zero.

4.2.5 Numerical example

Some of the techniques developed in previous sections will now be illustrated by means of an example.

This example is taken from Sridhar and Lindorff [9]. The plant is

$$\dot{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} x$$

This plant is both observable and controllable with $\nu_o = 2$ and $\nu_c = 2$. Hence Brasch and Pearson as well as Kimura would require a first-order compensator for complete pole assignment. However, (4-15) indicates that it may be possible to arbitrarily assign all four system poles using only constant output feedback, although Sridhar and Lindorff state that only two poles can be assigned thereby (they employ a restrictive unity-rank feedback matrix). To test this possibility, the desired closed-loop poles were taken as -1, -2, -3 and -5, the values chosen by Sridhar and Lindorff. Then the error measure J_1 was minimized with weights $S_i = 1$, $i = 1,\ldots,4$.

After 20 iterations of the Davidon-Fletcher-Powell algorithm, J_1 was reduced to 0.2918 E-6, with

$$F \approx \begin{bmatrix} 5.4 & -1.8 \\ -10.7 & 1.9 \end{bmatrix}$$

and the closed-loop poles were within 10^{-4} of the desired values. Thus exact pole assignment has in effect been achieved using only constant output feedback.

4.3 OTHER POSSIBLE DESIGN OBJECTIVES

By using a compensator order higher than that required for exact pole placement, there may be extra degrees of design freedom that can be utilized to attain other design objectives by minimizing a second performance index $J_2(F)$ while satisfying the constraint $J_1(F) = 0$. This may be achieved by minimizing the augmented performance index

$$J(F) = \gamma J_1(F) + J_2(F), \quad \gamma \geq 0 \tag{4-26}$$

and letting the penalty weight $\gamma \to \infty$.

It is interesting to note that for finite values of γ, lower values of $J_2(F)$ may be obtained although the constraint $J_1(F) = 0$ will not be satisfied exactly, i.e. a trade-off occurs between minimizing $J_2(F)$ and exact pole placement. This possibility is discussed further in the sequel.

4.3.1 Pole placement with minimum quadratic performance index

Several forms for $J_2(F)$ are discussed in this and subsequent chapters. In this section, $J_2(F)$ is assumed to be a quadratic performance index similar to that discussed in Chapter 2. The reason for choosing this particular performance index is because placement of the closed-loop poles at the desired locations does not automatically guarantee a good system response. The response also depends on the locations of the system zeros. Thus, it may be necessary in some way to weight the system response as well. Also, from a practical standpoint, it is important to accomplish a design that meets the specifications without requiring excessive control effort. From §2.2, $J_2(F)$ is given by

$$J_2(F) = E\{\tfrac{1}{2}\int_0^\infty \bar{x}'(t)Q\bar{x}(t) + u'(t)Ru(t) + \mathring{z}'(t)R_c\mathring{z}(t)dt\} \qquad (4\text{-}27)$$

where $Q \geqslant 0$, $R > 0$, $R_c > 0$ and with the assumption that the initial system state $\bar{x}_o$ is a random variable with known covariance, $\bar{X}_o$.

In view of the foregoing considerations, the new design problem may be formulated as follows.

Problem statement (S2)

Find the matrix F of the compensator parameters which minimizes the composite performance index

$$J(F) = \gamma J_1(F) + J_2(F) , \quad \gamma \geq 0 \tag{4-28}$$

subject to the constraint (4-5). J_2 is given by (4-27).

The minimization of $J(F)$ may be carried out using a gradient-type algorithm such as that of DFP since expressions for J_2, $\partial J_2/\partial F$ have been obtained in Chapter 2, §2.2. The relevant results are summarized below:

$$\left.\begin{array}{l}
J = \gamma J_1 + J_2 \\
\text{where} \\
J_1 = h'h \\
J_2 = \tfrac{1}{2}\,\mathrm{tr}\{K\bar{X}_o\} \\
\text{and} \\
\dfrac{\partial J}{\partial F} = \gamma \dfrac{\partial J_1}{\partial F} + \dfrac{\partial J_2}{\partial F} \\
\dfrac{\partial J_1}{\partial F} = 2\,Nh \\
\dfrac{\partial J_2}{\partial F} = \bar{R}F\bar{C}L\bar{C}' + \bar{B}'KL\bar{C} \\
\text{K, L are symmetric, positive definite solutions of the matrix equations} \\
K\bar{A}_o + \bar{A}_o'K + \bar{Q} + \bar{C}'F'\bar{R}F\bar{C} = 0 \\
L\bar{A}_o' + \bar{A}_oL + \bar{X}_o = 0 \\
\bar{R} \triangleq \begin{bmatrix} R & 0_{m,p} \\ 0_{p,m} & R_c \end{bmatrix}
\end{array}\right\} \tag{4-29}$$

h, N are given in §4.2.3.

Remarks

(R2). It is clear from remark (R1) in Chapter 2 that expressions for J_2 and its gradient with respect to F only hold when (4-5) is stable. Hence, the minimization algorithm being employed must be primed with initial guess of F that stabilizes (4-5). If $\overline{A}_o$ is not stable, then a procedure such as that described in §2.3 or §4.2 may be used to find a stabilizing value of F, if one exists. Unfortunately, the general conditions for the existence of a stabilizing F are not known at present. Only sufficient conditions such as that of Li [23] are currently available.

(R3). F must also be constrained to be a stabilizing matrix throughout the computation. Since the coefficient of the characteristic polynomial are computed at every iteration, this can be easily accomplished using a Hurwitz test.

(R4). When the system is subject to appreciable amounts of noise, the performance index (3-6) for the stationary stochastic regulator problem considered in §3.2 may be adopted for $J_2(F)$ instead of the form (4-27).

The problem posed in (S2) enables any one of several design objectives to be attained by suitably choosing the weighting matrics Q, $\overline{R}$, the constants S_i, $i = 1,\ldots,l$ and γ. These possibilities include:

(i) When the order of the compensator is greater than that required for exact placement of all $n + p$ poles, the extra degrees of design freedom may be used to minimize the control effort. This may be accomplished using what amounts to a penalty function technique by setting $Q = 0$

and letting the penalty weights $\gamma \to \infty$, while $\bar{R}$ is chosen as appropriate. A numerical example illustrating this case is presented in §4.3.2. Alternatively, a gradient projection algorithm [24] may be used to directly minimize the cost $J_2(F)$ subject to the constraint $J_1(F) = 0$.

(ii) If p is not sufficiently high to result in the exact placement of all $n+p$ poles, then make Q, $\bar{R}$, S_i, all i and γ finite and non-vanishing. The optimal F then yields a design that achieves a compromise between pole placement, good transient response and minimal control effort.

4.3.2 Example

The plant equations are

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x$$

This plant is also both observable and controllable, and has open-loop poles at 0, -1 and -5. Inequality (4-15) implies that complete placement of all the closed-loop poles requires at least a first-order compensator. Now from §4.2.2.1, a first-order compensator has five independent parameters whereas the closed-loop system has only four poles. Thus there is perhaps an extra degree of design freedom which may be employed to minimize the control effort required for exact pole placement.

Suppose that the desired closed-loop poles are $-1 \pm j2$, -2 and -4, and that

$$E\{\bar{x}(0)\bar{x}'(0)\} = \bar{X}_o = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then choose Q = 0 and let

$$\bar{R} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$S_i = 1, \quad i = 1, \ldots, 3$$

$$\gamma = 1000$$

Now it is intuitively obvious that an optimal *first-order* compensator should be separately controllable by each of its inputs; otherwise, the compensator would not be using all the plant outputs and therefore is unlikely to be optimal. Hence for this problem, P and N are assumed to have the canonical forms (C1) and (C2) respectively (see §2.2.2). Then, minimization using the Davidon-Fletcher-Powell algorithm yields the following results:

$$F = \begin{bmatrix} -19.600 & -7.999 & -0.778 \\ 1.000 & 0.501 & -2.000 \end{bmatrix}$$

$$J_1 = 0.00705$$

$$J_2 = 104.3$$

Hence the required compensator is

$$\dot{z} = [-2.000]\,z + [\;1.000 \quad 0.501]\,y$$

$$u = [-0.778]\,z + [-19.600 \quad -7.999]\,y$$

and the corresponding closed-loop poles are found to be $-1.001 \pm j1.9999$, -2.001 and -3.9999. Thus, exact pole placement has virtually been achieved while minimizing the mean square control effort.

4.4 POLE PLACEMENT IN PRESCRIBED REGIONS OF THE COMPLEX PLANE

It is clear from the above discussion that all the poles of a controllable and observable system may be assigned arbitrarily using output feedback in a dynamic compensator of sufficiently high order. However, in practice the exact positioning of the closed-loop poles is perhaps only of minor importance; it may suffice to position them within a prescribed region of the complex plane, see e.g. [26]. Moreover, it seems likely that this latter objective might usually be attained with a compensator of lower order.

Fortmann [27] has given an algorithm for finding a constant output feedback matrix that stabilizes a given linear system. By means of transformations, the algorithm can also produce designs with the system poles constrained to be in some particular regions of the complex plane. Although by working with the Hurwitz determinants, the approach in [27] has the advantage that the system poles do not have to be computed explicitly, it is limited in the form of the regions into which the poles can be positioned.

McBrinn and Roy [28] have given an algorithm for system stabilization that works directly with the closed-loop poles. In this section, their approach is improved and extended to treat the more general problem of positioning the closed-loop poles in an arbitrary region of the complex plane.

4.4.1 Problem formulation

Consider the system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \qquad (4\text{-}30)$$

where again, $x(t) \varepsilon R^n$, $y(t) \varepsilon R^r$ and $u(t) \varepsilon R^m$. A, B, C are constant matrices of appropriate dimensions.

The problem is to find a constant feedback law

$$u(t) = Fy(t) \qquad (4\text{-}31)$$

such that all the closed-loop system poles are in a region R of the complex plane defined by

$$h(\sigma, \omega) \leqslant 0 \qquad (4\text{-}32)$$

where $s = \sigma + j\omega$, $h(\cdot,\cdot)$ is a continuous and differentiable function. See Fig. 4.3 for an arbitrary function of $h(\cdot,\cdot)$.

Note that there is no loss of generality in assuming constant output feedback for it has been shown in §2.2.1 that the design of a dynamic compensator given by (4-4) can be reduced mathematically to the form described by (4-30) and (4-31). Also, extension to problems to include more than one constraint of the form (4-32) is straightforward.

Unfortunately, there is no known way of establishing the existence of a solution to the problem that has just been posed. One can only attempt to find a solution using an algorithm such as that described in the sequel.

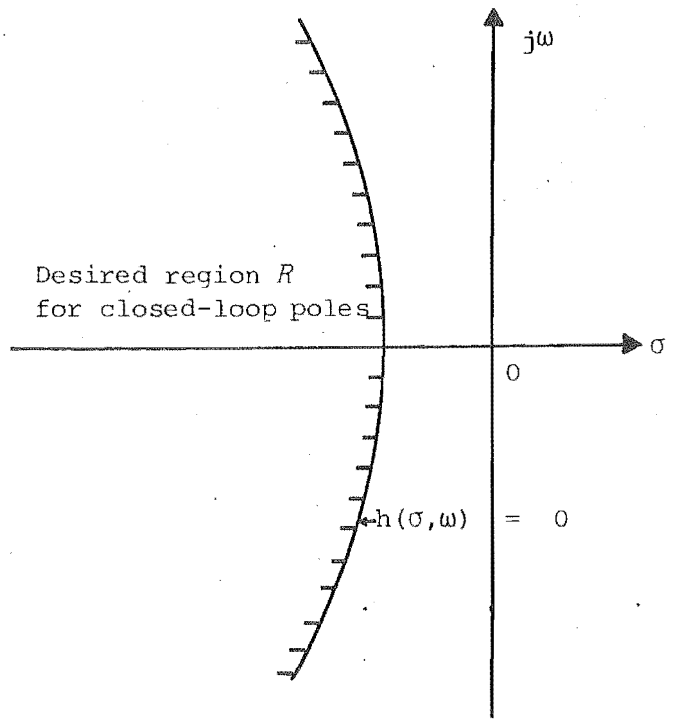

Fig. 4.3 An arbitrary function of h(σ,ω) in the complex plane s.

4.4.2 Solution technique

From (4-30) and (4-31), the closed-loop system is

$$\dot{x} = (A + BFC)x \triangleq \bar{A}_o x \tag{4-33}$$

and its poles are the roots $\lambda_i \triangleq \alpha_i + j\beta_i$, $i = 1,2,\ldots,n$, of the characteristic equation

$$\Delta(\lambda) \triangleq \det(\lambda I - \bar{A}_o) \triangleq \lambda^n + a_1\lambda^{n-1} + \ldots + a_n = 0 \tag{4-34}$$

The coefficients a_i, $i = 1,2,\ldots,n$, may be computed readily using the algorithm of Danilevsky [22]. Then (4-34) can be solved using a standard computer subroutine such as POLRT [29].

For a trial value of F, some closed-loop poles may lie outside the prescribed region R, and the problem is to modify F such that these undesirable poles move towards R. Since complex poles occur in conjugate pairs, and assuming that R is symmetric about the real axis, it is sufficient to consider those q undesirable poles that satisfy

$$\beta_i \geqslant 0 \qquad i = 1,2,\ldots,q, \tag{4-35}$$

i.e. that lie above or on the real axis. The aim of the design is to make q = 0.

Now considering first-order terms only, this implies that the increment δF must be chosen so that

$$h(\alpha_i,\beta_i) + \left[\frac{\partial h(\alpha_i,\beta_i)}{\partial F}\right]' \delta F = 0, \qquad i = 1,2,\ldots,q \tag{4-36}$$

where δF and $\frac{\partial h}{\partial F}$ are stored as $rm \times 1$ column vectors. (4-36) can be written as

$$H + N\,\delta F = 0 \tag{4-37}$$

where

$$H = \text{col } [h(\alpha_1, \beta_1), h(\alpha_2, \beta_2), \ldots, h(\alpha_q, \beta_q)]$$

$$N = \left[\frac{\partial h(\alpha_1, \beta_1)}{\partial F}, \frac{\partial h(\alpha_2, \beta_2)}{\partial F}, \ldots, \frac{\partial h(\alpha_q, \beta_q)}{\partial F}\right]'.$$

Since N is, in general, a rectangular matrix and moreover may not have full rank, the 'solution' of eqn. (4-37) will be taken as

$$\delta F = -N^\dagger H \tag{4-38}$$

where $N^\dagger$ is the pseudoinverse of N [30]. The solution (4-38) satisfies (4-37) exactly only when rank (N) = q. When rank (N) < q, (4-38) 'satisfies' (4-37) only in a minimum square error sense. It may be noted that (4-38) is a generalization of the constraint restoration step proposed by Rosen [24] to find feasible solutions for non-linear programming problems.

In practice, the increment (4-38) may on occasion be too large for the first-order expansion in (4-36) to be valid, and this may lead to computational instability. Hence, provision is made in the algorithm described below for repeatedly halving δF until a favourable F is obtained.

It remains to find computable expressions for $\partial h(\alpha_i, \beta_i)/\partial F$. First, observe that

$$\frac{\partial h(\alpha_i, \beta_i)}{\partial F} = \frac{\partial h(\alpha_i, \beta_i)}{\partial \alpha_i} \cdot \frac{\partial \alpha_i}{\partial F} + \frac{\partial h(\alpha_i, \beta_i)}{\partial \beta_i} \cdot \frac{\partial \beta_i}{\partial F} \tag{4-39}$$

Now $\partial\alpha_i/\partial F$, $\partial\beta_i/\partial F$ are respectively the real and imaginary parts of $\partial\lambda_i/\partial F$ which is given by

$$\frac{\partial \lambda_i}{\partial F} = \frac{B' V_i W_i ' C'}{V_i ' W_i} \qquad (4\text{-}40)$$

where V_i, W_i are the row and column eigenvectors of $\bar{A}_o$ corresponding to λ_i. Equation (4-40) is derived using a result of Faddeev and Faddeeva [22]. Note that V_i and W_i may also be computed using the algorithm of Danilevsky referred to previously. Alternatively, $\partial\lambda_i/\partial F$ may be obtained using the results of previous sections since

$$\frac{\partial \lambda_i}{\partial F} = - \left[\frac{1}{\partial\Delta/\partial\lambda} \cdot \frac{\partial \Delta}{\partial F} \right]_{\lambda = \lambda_i} \qquad (4\text{-}41)$$

where the RHS is known for

$$\frac{\partial \Delta}{\partial \lambda} = \sum_{j=1}^{n-1} j \, a_{n-j} \lambda^{j-1} \qquad (4\text{-}42)$$

and $\partial\Delta/\partial F$ is also known from §4.2.3.2.

4.4.3 The algorithm

At this point, sufficient expressions for solving the problem have been derived and the following algorithm is proposed.

Step (i): Choose an arbitrary value of F to initiate the search.

Step (ii): Determine the closed-loop poles λ_i using the Danilevsky algorithm in conjunction with a root-finding subroutine. Find and store all λ_i, $i = 1,2,\ldots,q$ that satisfy

$$h(\alpha_i, \beta_i) > 0$$

$$\beta_i \geqslant 0$$

If none of the λ_i lie in this region, the required design has been achieved and computation is terminated.

Step (iii): Compute

$$J \triangleq \sum_{i=1}^{q} \{h(\alpha_i,\beta_i)\}^2 = H'H$$

(This function is introduced to monitor progress towards the solution. Notice that any undesirable poles that may appear are at once included in the evaluation of J.) Then proceed to step (iv) if on the first iteration, or if the new value of J is less than its previous lowest value. Otherwise, halve the increment δF and return to step (ii).

Step (iv): Compute $\partial h(\alpha_i,\beta_i)/\partial F$ for $i = 1,2,\ldots,q$, and so construct N to obtain δF using eqns. (4-38) - (4-40). Stop with a flag indicating a local minimum of J if $\|\delta F\|$ is smaller than a prescribed threshold. Otherwise update F and return to step (ii).

The algorithm in effect employs an increment

$$\delta F = -\theta N^{\dagger} H \qquad 0 < \theta \leqslant 1 \tag{4-43}$$

where θ is a convergence factor. For sufficiently small θ, there will be no change in q and the corresponding change in J will be given with sufficient accuracy by

$$\delta J = \left(\frac{\partial J}{\partial F}\right)' \delta F = -2\theta H' N N^{\dagger} H . \tag{4-44}$$

When rank (N) = q, it can be shown that $NN^{\dagger}$ is positive definite. Hence $\partial J < 0$ unless $H = 0$, that is when a solution has been obtained. When rank $(N) < q$, $NN^{\dagger}$ is only positive semidefinite. Hence $\delta J \leqslant 0$, where the equality sign may occur for $H \neq 0$. In fact it can be shown that $\delta J = 0$ whenever $N'H = \frac{1}{2}\frac{\partial J}{\partial F} = 0$, i.e. whenever a local minimum of J has been reached.

Thus a reduction in the value of J will be achieved at every iteration of the algorithm unless a local minimum of J has been reached. Hence, unless the latter occurs first, it is clear that J will be eventually reduced to zero, thus yielding a solution to the problem. If, however, a local (non-zero) minimum of J is reached, the algorithm should be restarted with a different initial value of F. Repeated failure to reduce J to zero would indicate the absence of a solution.

Unlike in the algorithm of McBrinn and Roy [28], here one attempts to move all the undesirable poles simultaneously. Moreover, in place of the expensive gradient search used by them, the new algorithm employs a definite step (4-38) which is usually found to be favourable (the increment-halving feature in step (iii) is only rarely called upon in practice). For these reasons it is reasonable to presume that both the number of iterations and computation time per iteration would, in most cases, be significantly lower for the new algorithm.

4.4.4 Numerical examples

Three examples are included here to illustrate the usefulness of the techniques developed.

Example 1

A seven-state, two-output, single-input model of a Saturn V booster [28] can be represented by

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & -0.65 & -0.002 & 2.6 & 0 \\ -0.014 & 1 & -0.041 & 0.0002 & -0.015 & -0.033 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -45 & -0.13 & 255 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -50 & -10 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} x(t) .$$

The open-loop poles are 0.014, 0.420, -0.475, $-5 \pm j5$ and $-0.065 \pm j\,6.708$. It is required to find a feedback matrix F such that all the closed-loop poles satisfy

$$h(\alpha,\beta) = \alpha + 0.07 \leqslant 0 . \tag{4-45}$$

Starting with F = 0, the constraint (4-45) was satisfied after three iterations of the algorithm with

$$F = [20.31,\ 16.56] .$$

The corresponding closed-loop poles are -0.098, $-0.125 \pm j\,.497$, $-4.841 \pm j\,5.433$ and $-0.070 \pm j\,6.204$. Computation time was about 30 seconds on an EAI 640 minicomputer.

Example 2

The system state equation is

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -2.93 & -4.75 & -0.78 \\ 0.086 & 0 & -0.11 & -1.00 \\ 0 & -0.042 & 2.59 & -0.39 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0 & -3.91 \\ 0.035 & 0 \\ -2.53 & 0.31 \end{bmatrix} u(t)$$

and the open-loop poles are -0.041, -2.989, $-0.200 \pm j\,1.640$. Only the first three state variables are available for feedback. It is required to position the closed-loop poles to the left of the parabolic boundary

$$h(\alpha,\beta) = \alpha + 0.5\,\beta^2 + 0.3 = 0$$

The starting F was again chosen to be the null matrix, and the required design was obtained after three iterations with

$$F = \begin{bmatrix} -0.102 & -0.114 & 0.987 \\ -0.131 & 0.116 & 0.971 \end{bmatrix}$$

The corresponding closed-loop poles are $-0.436 \pm j\,0.416$ and $-2.123 \pm j\,1.359$.

Example 3

The open-loop system is described by

$$\dot{x} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & -2 & -2 \\ 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x$$

and has poles at 1, -1 and -2. The problem is to find a feedback matrix F such that the closed-loop poles have a real part less than -0.4 and a damping ratio greater than $1/\sqrt{5}$.

Here the closed-loop poles must satisfy the two constraints

$$h_1(\alpha,\beta) = \alpha + 0.4 \leqslant 0$$

$$h_2(\alpha,\beta) = \alpha + 0.5\,|\beta| \leqslant 0.$$

The algorithm was extended in a straightforward fashion to deal with this case. Again the constraints were satisfied after three iterations commencing with F = 0. The final design was

$$F = \begin{bmatrix} -0.184 & 0.180 \\ -0.187 & 0.254 \end{bmatrix}$$

with closed-loop poles at -0.702, and -0.528 ± j 0.841 .

4.5 CONCLUSIONS

The problem of designing dynamic compensators for linear multivariable systems has been formulated as a static optimization problem in parameters defining the compensator, and a suitable solution algorithm has been derived. Unlike in the work of Brasch and Pearson [8] and others, no restrictions have been imposed on the ranks of the compensator matrices. Consequently, arbitrary pole placement may be achieved using compensators of lower order than required by them. However, an explicit expression for the minimum compensator order for exact pole placement has not been obtained; only a lower bound on the required compensator order has been found. Further research is needed to resolve this aspect of the problem.

The more general problem of achieving either exact or approximate pole placement while minimizing a quadratic performance index in state and control variables has also been considered. This problem formulation permits the simultaneous attainment of several design objectives such as obtaining pole placement while either optimizing the system response or minimizing the mean square control effort. It is, of course, possible to consider other types of performance indices. This will be described in Chapters 5 and 6.

Consideration is also given to the design of output feedback control systems with poles constrained to lie in prescribed regions of the complex plane. A method has been presented which is computationally efficient as has been borne out by the numerical examples. It is also anticipated that this approach will yield a practical design having simpler structures (e.g., lower compensator order) than those required for exact pole placement.

APPENDIX A

The problem formulation and solution when some of the desired closed-loop poles are equal will be considered in this Appendix.

Suppose that $\eta + 1$ of the preassigned poles are equal, say

$$\lambda_\mu = \lambda_{\mu+1} = \ldots = \lambda_{\mu+\eta} .$$

Then the condition for exact pole placement is

$$\left.\frac{d^j \Delta(\lambda)}{d\lambda^j}\right|_{\lambda = \lambda_\mu} = 0, \quad j = 0, 1, \ldots, \eta \qquad (4\text{-}46)$$

Hence it would be appropriate to modify the performance index (4-12) to

$$J_1 = \tfrac{1}{2}\left\{ \sum_{i=1}^{\mu-1} s_i |\Delta(\lambda_i)|^2 + \sum_{i=\mu}^{\mu+\eta} s_i \left| \frac{d^{i-\mu}\Delta(\lambda_i)}{d\lambda_i^{\,i-\mu}} \right|^2 + \sum_{i=\mu+\eta+1}^{q} s_i |\Delta(\lambda_i)|^2 \right\} \qquad (4\text{-}47)$$

If the DFP minimization algorithm is to be used, then $\partial J_1/\partial F$ can be obtained by noting that

$$\frac{\partial}{\partial F} \frac{d^j \Delta(\lambda)}{d\lambda^j} = \frac{d^j}{d\lambda^j} \frac{\partial \Delta(\lambda)}{\partial F} = -B' \frac{d^j \Phi(\lambda)}{d\lambda^j} \bar{C}' \qquad (4\text{-}48)$$

Now $d^j\Phi(\lambda)/d\lambda^j$ is readily computable from (4-21) after the matrix coefficients $\phi_1, \phi_2, \ldots, \phi_{n+p-1}$ have been computed using the Faddeev algorithm. The computation of $\partial J_1/\partial F$ then becomes straightforward.

Finally, the cost component J_2 does not depend on the values of the desired closed-loop poles. Hence, there is no change in the methods for computing J_2 and $\partial J_2/\partial F$.

APPENDIX B

Description of the Marquardt algorithm

Although the Marquardt algorithm remains one of the most powerful techniques for minimizing of functions having sum-of-squares form, it has not been used extensively in the control literature. A brief description of the algorithm is therefore appropriate.

The problem is: find the m-vector F so that the function

$$f(F) = H'H$$

is minimized. H is a n-dimensional vector function of F.

The Gauss-Newton updating formula for this problem is [1]

$$\delta F = \begin{cases} -(NN')^{-1} NH, & \text{if } m \leqslant n \\ -N(N'N)^{-1} H, & \text{if } m > n \end{cases} \qquad (4\text{-}49)$$

where

$$N'_{ij} \triangleq \frac{\partial H_i}{\partial F_j}$$

Consider the case $m \leqslant n$. It has been discovered [1] that better convergence may be obtained if the Marquardt algorithm [2] is adopted, i.e., modify (4-49) to

$$\delta F = -(NN' + \lambda I_m)^{-1} NH \qquad (4\text{-}50)$$

However, in the present research program, further modifications of (4-50) suggested by Box and Jenkins [3] has been adopted. This involves the normalization of N, H so as to avoid ill-conditioning of the matrix inversion. First set $A = NN'$, $C = -NH$; then the complete algorithm is as follows:

Step (i): Set a vector d where

$$d_i = \sqrt{A_{ii}}\,, \qquad i = 1,\ldots,m$$

Step (ii): Compute A^* where

$$\left.\begin{array}{ll} A^*_{ii} = 1 + \pi\,, & \pi \text{ is an arbitrary constant (see below)} \\ A^*_{ij} = A_{ij}/d_i d_j\,, & i \neq j \end{array}\right\} \begin{array}{l} i=1,\ldots,m \\ j=1,\ldots,m \end{array}$$

Step (iii): Compute C^* where

$$C_i{}^* = C_i/d_i \qquad i = 1,\ldots,m$$

Step (iv): Then

$$\delta F^* = (A^*)^{-1}C^*$$

and

$$\delta F_j = \delta F^*/\overset{d_j}{\cancel{C_j}} \qquad j = 1,\ldots,m$$

δF obtained in step (iv) is acceptable if the function to be minimized, $f(F)$, decreases, otherwise multiply π by a factor k (say 2) and return to (ii).

The starting π is (arbitrarily) chosen to be 0.1 for the examples shown in this and the next chapter.

[1] Dixon, L.C.W., Nonlinear programming. The English Universities Press Ltd, 1972.

[2] Marquardt, D.W., "An algorithm for least squares estimation of nonlinear parameters," SIAM J., vol. 11, pp.431-441, 1963.

[3] Box, G.E.P. and Jenkins, G.M., Time-series analysis, forecasting and control. Holden-Day, San Francisco, 1971.

REFERENCES

[1] Davison, E.J., "On pole assignment in multivariable linear systems," IEEE Trans. Automat. Contr., vol. AC-14, pp.747-748, Dec. 1968.

[2] Wonham, H.W., "On pole assignment in multi-input controllable linear systems," IEEE Trans. Automat. Contr., vol. AC-12, pp.660-665, Dec. 1967.

[3] Davison, E.J. and Chatterjee, R., "A note on pole assignment in linear systems with incomplete state feedback," IEEE Trans. Automat. Contr., vol. AC-16, pp.98-99, Feb. 1971.

[4] Davison, E.J. and Chow, S.G., "An algorithm for the assignment of closed-loop poles using output feedback in large linear multivariable systems," IEEE Trans. Automat. Contr., vol. AC-18, Feb. 1973.

[5] Davison, E.J. and Wang, S.H., "On pole assignment in linear multivariable systems using output feedback," IEEE Trans. Automat. Contr., vol. AC-20, pp.516-518, Aug. 1975.

[6] Kimura, H., "Pole assignment by gain output feedback," IEEE Trans. Automat. Contr., vol. AC-20, pp.509-516, Aug. 1975.

[7] Ahmari, R. and Vacroux, A.G., "On the pole assignment in linear systems with fixed order compensators," Int. J. Contr., vol. 17, pp.397-404, Feb. 1973.

[8] Brasch, F.M., Jr. and Pearson, J.B., "Pole placement using dynamic compensators," IEEE Trans. Automat. Contr., vol. AC-15, pp.34-43, Feb. 1970.

[9] Sridhar, B. and Lindorff, D.P., "Pole-placement with constant gain output feedback," Int. J. Contr., vol. 18, pp.993-1003, Nov. 1973.

[10] Fallside, F. and Seraji, H., "Direct design procedure for multivariable feedback systems," Proc. I.E.E., vol. 118, pp.797-801, June 1971.

[11] Fallside, F. and Seraji, H., "Pole-shifting procedure for multivariable systems using output feedback," Proc. I.E.E., vol. 118, pp.1648-1654, Nov. 1971.

[12] Fallside, F. and Seraji, H., "Design of multivariable systems using unity-rank feedback," Int. J. Contr., vol. 17, pp.351-364, Feb. 1973.

[13] Sirisena, H.R. and Choi, S.C., "Optimal pole placement in linear multivariable systems using dynamic output feedback," Int. J. Contr., vol. 21, pp.661-671, April 1975.

[14] Patel, R.V., "Pole assignment by means of unrestricted-rank output feedback," Proc. I.E.E., vol. 121, pp.874-878, Aug. 1974.

[15] Maki, M.C. and Van de Vegte, J., "Optimization of multiple-input systems with assigned poles," IEEE Trans. Autom. Contr. vol. AC-19, pp.130-133, Apr. 1974.

[16] Ahmari, R. and Vacroux, A.G., "Approximate pole placement in linear multivariable systems using dynamic compensators," Int. J. Contr., vol. 18, pp.1329-1336, Dec. 1973.

[17] Sirisena, H.R. and Choi, S.S., "Pole placement in prescribed regions of the complex plane using output feedback," IEEE Trans. Automat. Contr., vol. AC-20, pp.810-812, Dec. 1975.

[18] Fletcher, R. and Powell, M.J.D., "A rapidly convergent descent method for minimization," British Computer J., pp.163-168, June 1963.

[19] Dixon, L.C.W., Nonlinear programming. The English Universities Press Ltd, 1972.

[20] Marquardt, D.W., "An algorithm for least squares estimation of nonlinear parameters," SIAM J. vol. 11, pp.431-441, 1963.

[21] Gantmacher, F.R., The Theory of Matrices, vol. I. New York: Chelsea, 1959.

[22] Faddeev, D.K. and Faddeeva, V.N., Computational Methods of Linear Algebra. San Francisco, Freeman and Co., 1963.

[23] Li, M.T., "On output feedback stabilizability of linear systems," IEEE Trans. Automat. Contr., vol. AC-17, pp.408-410, June 1972.

[24] Rosen, J.B., "The gradient projection method for nonlinear programming, II," SIAM J., vol. 9, 1961.

[25] Davison, E.J., "On pole assignment in linear systems with incomplete state feedback," IEEE Trans. Automat. Contr., vol. AC-15, pp.348-351, June 1970.

[26] Chen, C.T., Analysis and Synthesis of Linear Control Systems. HRW series in Electrical Engineering, Electronics and Systems 1975.

[27] Fortmann, T.E., "Stabilization of multivariable system with constant gain output feedback," in Proc. JACC, pp.294-301, 1973.

[28] McBrinn, D.E. and Roy, R.J., "Stabilization of linear multivariable systems by output feedback," IEEE Trans. Automat. Contr., vol. AC-17, pp.243-245, April 1972.

[29] IBM Application Program, System-360 Scientific Subroutine Package (360A-CM-03X), Version III.

[30] Penrose, R.A., "A generalized inverse for matrices," Proc. Cambridge Phil. Soc., vol. 51, pp.406-413, 1955.

CHAPTER 5

POLE PLACEMENT IN OUTPUT FEEDBACK CONTROL SYSTEMS FOR MINIMUM SENSITIVITY TO PLANT PARAMETER VARIATIONS

5.1 INTRODUCTION

In this chapter an extension of the pole-placement algorithm of Chapter 4 to the design of output feedback laws for pole placement that, in some sense, have minimum sensitivity to plant parameter variations is described. The motivation for this study is obvious. The linear differential description of the multivariable plant is usually an approximate model of a physical system such that almost all the parameters involved in the differential description of the dynamic system are subject to variation and uncertainty. These parameter variations may be small or large. Therefore, it is necessary to develop design techniques such that the effect of these variations on the closed-loop poles is reduced or minimized.

One of the earliest investigations in this area was that by Morgan [2] who derived several useful expressions for the sensitivity of the closed-loop poles to small parameter changes. The results of [2] have also been extended and used in [3] in a sensitivity study of system response subject to small parameter changes. However, the formulae obtained in [3] are so complex that their effectiveness in controller design is doubtful. More recently, Tzafestas and Paraskevopoulos [4] have given an algorithm for designing constant output feedback law with decreased sensitivity to plant parameter variations. However, their feedback law contains the actual deviations of the plant parameters from their nominal values, and it is unrealistic to presume that these deviations will be known in practice.

In this chapter, two different problem formulations for pole placement with minimum sensitivity are considered. In Problem A, §5.2, the deviations of the plant parameters are assumed to be small (but unknown) compared to their nominal values. In Problem B, §5.3, the plant parameter variations are permitted to be comparable in magnitude to the nominal parameter values, and it is assumed that the probability distributions of the parameters about their nominal values are given.

For clarity, the treatment is confined to constant output feedback laws, but the extension to the design of dynamic compensators as in Chapter 4 [1] is straightforward. Similarly, although perturbations in only a single plant parameter are considered, there is no difficulty in treating perturbations in more than one parameter. The work described in this chapter has also appeared in a paper by Sirisena and Choi [10].

5.2 DESIGN TECHNIQUE FOR SMALL PARAMETER VARIATIONS - PROBLEM A

5.2.1 The system

The plant is described by the equations

$$\left.\begin{aligned} \dot{x} &= A(\mu)x + B(\mu)u \\ y &= C(\mu)x \end{aligned}\right\} \qquad (5\text{-}1)$$

where x, u and y are vectors of dimensions n, m and r respectively. The matrices A, B and C have appropriate dimensions and are assumed to be known functions of the parameter μ. The feedback laws considered are of the form

$$u = Fy \qquad (5\text{-}2)$$

where F is a constant $m \times r$ matrix.

The closed-loop system is therefore

$$\dot{x} = [A(\mu) + B(\mu)FC(\mu)]x \triangleq \overline{A}(\mu)x \qquad (5\text{-}3)$$

and the closed-loop poles λ_i, $i = 1,2,\ldots\ldots,n$, are the roots of the characteristic equation

$$\Delta(\lambda,\mu) \triangleq |\lambda I - \overline{A}(\mu)| = 0 \qquad (5\text{-}4)$$

5.2.2 Pole placement problem and the sensitivity function

Suppose the n desired closed-loop poles λ_i^d, $i = 1,\ldots,n$ are specified where in a manner similar to that described in §4.2.1, k of these poles are real while the remainder are complex and appear in conjugate pairs. Let the poles be ordered such that

$$\mathrm{Im}(\lambda_i^d) = 0 \quad , \qquad i = 1,\ldots,k$$

$$\mathrm{Im}(\lambda_i^d) > 0 \quad , \qquad i = k+1,\ldots,l$$

$$\mathrm{Im}(\lambda_i^d) < 0 \quad , \qquad i = l+1,\ldots,n$$

where, as before,

$$l = (n+k)/2$$

Assume that λ_i^d, all i, are distinct. Extension to the case of repeated poles is possible but tedious. (See e.g. Appendix A of Chapter 4). Also, there is actually no great loss of generality in making this assumption because the choice of the λ_i^d is at the designer's discretion.

Then, from §4.2.1 where it is shown that a reasonable performance index for the pole placement problem is given by

$$J_c = \sum_{i=1}^{l} |\Delta(\lambda_i^d, \mu_o)|^2 \tag{5-5}$$

where μ_o is the nominal value of μ. Clearly, $J_c \equiv 0$ if and only if exact pole placement has been achieved, i.e. $\lambda_i = \lambda_i^d$, all i.

Now consider the sensitivity aspect of the problem. In general, deviations of the plant parameter μ from its nominal value will cause the actual system poles λ_i to deviate from their desired locations λ_i^d. The sensitivity of the system poles to small perturbations in μ may be measured by the sensitivity function

$$J_s = \sum_{i=1}^{l} s_i^2 \left|\frac{\partial\lambda_i}{\partial\mu}\right|^2_{\mu=\mu_o} \tag{5-6}$$

where the summation is again over only the l poles with non-negative imaginary parts since

$$\left|\frac{\partial \lambda_i^*}{\partial \mu}\right| = \left|\frac{\partial \lambda_i}{\partial \mu}\right| .$$

The arbitrary weighting constants S_i in (5-6) are introduced to give design flexibility. It is convenient to express J_s in terms of the system characteristic polynomial. This may be done by using

$$\frac{\partial \lambda_i}{\partial \mu} = -\left[\frac{\partial \Delta(\lambda,\mu)}{\partial \mu} \Big/ \frac{\partial \Delta(\lambda,\mu)}{\partial \lambda}\right]_{\lambda=\lambda_i} \tag{5-7}$$

Now since the exact pole placement constraint is to be satisfied, i.e. $\lambda_i = \lambda_i^d$, all i, it follows from (5-7) that

$$\left[\frac{\partial \lambda_i}{\partial \mu}\right]_{\mu=\mu_o} = -\frac{\partial \Delta(\lambda_i^d,\mu_o)}{\partial \mu_o} \Big/ \left[\frac{\partial \Delta_d(\lambda)}{\partial \lambda}\right]_{\lambda=\lambda_i^d}$$

where $\Delta_d(\lambda) \triangleq \prod_{i=1}^{n} (\lambda-\lambda_i^d)$ is the desired characteristic polynomial. Hence J_s may be written as

$$J_s = \sum_{i=1}^{l} S_i^2 \left|\frac{\partial \Delta(\lambda_i^d,\mu_o)}{\partial \mu_o}\right|^2 \Big/ \left|\frac{\partial \Delta_d(\lambda)}{\partial \lambda}\right|^2_{\lambda=\lambda_i^d} \tag{5-8}$$

Moreover, the constants

$$\left|\frac{\partial \Delta_d(\lambda)}{\partial \lambda}\right|_{\lambda=\lambda_i^d} = \prod_{\substack{j=1\\ j\neq i}}^{n} \left|\lambda_i^d - \lambda_j^d\right|$$

can be precalculated and combined with the arbitrary constants S_i to give

$$J_s = \sum_{i=1}^{l} \overline{S}_i^2 \left|\frac{\partial \Delta(\lambda_i^d,\mu_o)}{\partial \mu_o}\right|^2 \tag{5-9}$$

where

$$\bar{S}_i \triangleq S_i \Big/ \prod_{\substack{j=1 \\ j\neq 1}}^{n} \left|\lambda_i^d - \lambda_j^d\right| .$$

5.2.3 Statement of Problem A

Problem A may now be stated as follows. Given the system (5-1), (5-2) and a set of desired poles λ_i^d, $i = 1,2,\ldots,n$, find the feedback matrix F that minimizes the sensitivity cost function J_s given by (5-6), or equivalently by (5-9), subject to $J_c = 0$.

Clearly Problem A can have a solution only if there exists at least one F that gives exact pole placement. Since F has mr elements, a necessary condition for exact pole placement to be possible with arbitrary λ_i^d is

$$mr \geqslant n \tag{5-10}$$

However, (5-10) is not a sufficient condition and in fact the necessary and sufficient conditions for pole placement using constant output feedback are not known at present, as has been explained in Chapter 4. Existence of a solution to Problem A can therefore only be determined by numerical experimentation with the actual system.

5.2.4 Algorithm for the solution of Problem A

5.2.4.1 Solution technique

Problem A is evidently a constrained minimization problem, and in general both the objective function J_s and the constraint function J_c are non-linear. A numerical solution may be obtained using standard techniques such as the gradient projection algorithm of Rosen [5] or the penalty function method. Rosen's algorithm is difficult to apply with non-linear constraints. On the other hand, numerical experimentation reveals that the penalty function approach is usually very satisfactory for the problem in hand. In this approach, an augmented cost function

$$J = J_s + \gamma^2 J_c \tag{5-11}$$

is minimized with respect to F using an unconstrained minimization algorithm. The violation of the constraint $J_c = 0$ at the minimum of J may be made arbitrarily small by making the penalty weight γ sufficiently large. The minimization of J may be performed using a quasi-Newton algorithm such as that of [6], or preferably an algorithm that exploits the sum of squares form of J, as was done in Chapter 4.

First, J is expressed in the form

$$J = h' \cdot h \tag{5-12}$$

where h is a real 2n-vector that is defined as

$$\begin{aligned} h = \Bigg[& \gamma\Delta(\lambda_i^d,\mu_o), \ldots, \gamma\Delta(\lambda_k^d,\mu_o),\ \gamma\,\mathrm{Re}\{\Delta(\lambda_{k+1}^d,\mu_o)\}, \gamma\,\mathrm{Im}\{\Delta(\lambda_{k+1}^d,\mu_o)\}, \\ & \ldots, \gamma\,\mathrm{Re}\{\Delta(\lambda_1^d,\mu_o)\}, \gamma\,\mathrm{Im}\{\Delta(\lambda_1^d,\mu_o)\},\ \bar{s}_1 \frac{\partial\Delta(\lambda_1^d,\mu_o)}{\partial\mu_o}, \ldots, \\ & \bar{s}_k \frac{\partial\Delta(\lambda_k^d,\mu_o)}{\partial\mu_o},\ \bar{s}_{k+1}\,\mathrm{Re}\left\{\frac{\partial\Delta(\lambda_{k+1},\mu_o)}{\partial\mu_o}\right\},\ \bar{s}_{k+1}\,\mathrm{Im}\left\{\frac{\partial\Delta(\lambda_{k+1}^d,\mu_o)}{\partial\mu_o}\right\}, \\ & \ldots, \bar{s}_1\,\mathrm{Re}\left\{\frac{\partial\Delta(\lambda_1^d,\mu_o)}{\partial\mu_o}\right\},\ \bar{s}_1\,\mathrm{Im}\left\{\frac{\partial\Delta(\lambda_1^d,\mu_o)}{\partial\mu_o}\right\} \Bigg]' . \end{aligned} \tag{5-13}$$

Now if F is stored as an mr × 1 column vector, and the mr $\times$ 2n matrix N is defined by

$$N = \frac{\partial}{\partial F} h', \tag{5-14}$$

then the well known Gauss-Newton iteration formula for minimizing J with respect to F is given by

$$\delta F = \begin{cases} -(NN')^{-1}Nh, & \text{if } mr \leqslant 2n \\ -N(N'N)^{-1}h, & \text{if } mr > 2n \end{cases} \tag{5-15}$$

In practice, better convergence is usually obtained using the modified version of formula (5-15) proposed by Marquardt [7] (see also Appendix B of Chapter 4).

Notice that in the Marquardt algorithm, each iteration step is independent of the previous one and hence there is no difficulty in changing the penalty weight γ during the computation if this is required. On the other hand, with the Davidon-Fletcher-Powell algorithm, the hessian matrix has to be reset each time γ is changed and this has an adverse effect on convergence.

In the next two sections, computable expressions for h and N are derived.

5.2.4.2 Computation of the vector h

Construction of the vector h requires the evaluation of $\Delta(\lambda,\mu_o)$ and $\partial\Delta(\lambda,\mu_o)/\partial\mu_o$ at the current value of F. The first step is to evaluate the nominal closed-loop matrix $\bar{A}(\mu_o)$ given by (5-3). Then the coefficients a_i of $\Delta(\lambda,\mu_o)$, where

$$\Delta(\lambda,\mu_o) \triangleq \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_n, \tag{5-16}$$

are computed using the following algorithm due to Faddeev [8]:

$$\left.\begin{aligned} a_i &= -\frac{1}{i}\,\mathrm{tr}\,[\phi_{i-1}\bar{A}'], \\ \phi_i &= \phi_{i-1}\bar{A}' + a_i I_n, \end{aligned}\right\} \quad i = 1,2,\ldots\ldots,n \tag{5-17}$$

where

$$\phi_o = I_n.$$

Of course, the algorithm of Davilevsky [9] may also be used to evaluate a_i as shown in §4.2.3.1.

Next, application of the chain rule for differentiation yields

$$\frac{\partial \Delta(\lambda,\mu_o)}{\partial \mu_o} = \left[\frac{\partial \Delta}{\partial \bar{A}} \odot \frac{\partial \bar{A}}{\partial \mu}\right]_{\mu=\mu_o} \tag{5-18}$$

where the matrix operation $\odot$ is defined by

$$X \odot Y = \sum_i \sum_j x_{ij} y_{ij}.$$

Now it can be easily shown that

$$\frac{\partial \Delta}{\partial \bar{A}} = -\Phi(\lambda,\mu) \tag{5-19}$$

where $\Phi(\lambda,\mu)$, as shown in §4.2.3.2, is the cofactor matrix of the determinant $|\lambda I_n - \bar{A}(\mu)|$ and can be expressed as

$$\Phi(\lambda,\mu) = \lambda^{n-1} I_n + \lambda^{n-2}\phi_1(\mu) + \ldots\ldots + \phi_{n-1}(\mu) \tag{5-20}$$

The matrix coefficients ϕ_i, $i = 1,2,\ldots\ldots,n-1$, are also generated by Faddeev's algorithm (5-17). Again it can be easily shown that

$$\bar{A}_\mu(\mu) = A_\mu(\mu) + B_\mu(\mu) F\, C(\mu) + B(\mu) F\, C_\mu(\mu) \tag{5-21}$$

where the subscript μ denotes differentiation with respect to

μ. Substitution from (5-19) and (5-21) in (5-18) then yields

$$\frac{\partial\Delta(\lambda,\mu_o)}{\partial\mu_o} = -\Phi(\lambda,\mu_o) \odot \{A_{\mu_o} + B_{\mu_o} FC(\mu_o) + B(\mu_o)FC_{\mu_o}\} \qquad (5\text{-}22)$$

where A_{μ_o}, B_{μ_o} and C_{μ_o} denote respectively the values of A_μ, B_μ and C_μ at $\mu = \mu_o$.

The elements of h can now all be obtained by repeated substitution of $\lambda = \lambda_i^d$, $i = 1,2,\ldots\ldots,1$, in (5-16) and (5-22).

5.2.4.3 Computation of the matrix N

The matrix N is composed of elements of the form $\frac{\partial}{\partial F}\Delta(\lambda_i^d,\mu_o)$ and $\frac{\partial}{\partial F}[\partial\Delta(\lambda_i^d,\mu_o)/\partial\mu_o]$, and expressions for these quantities will now be obtained.

Firstly, from §4.2.3.2, it is seen that

$$\frac{\partial\Delta(\lambda,\mu)}{\partial F} = -B'(\mu)\Phi(\lambda,\mu)C'(\mu) \qquad (5\text{-}23)$$

Next, by changing the order of differentiation and substituting from (5-23) it is also seen that

$$\frac{\partial}{\partial F}\left[\frac{\partial\Delta(\lambda,\mu)}{\partial\mu}\right] = \frac{\partial}{\partial\mu}\left[\frac{\partial\Delta(\lambda,\mu)}{\partial F}\right] = -\Big[B'_\mu(\mu)\Phi(\lambda,\mu)C'(\mu)$$

$$+ B'(\mu)\Phi_\mu(\lambda,\mu)C'(\mu) + B'(\mu)\Phi(\lambda,\mu)C'_\mu(\mu)\Big], \qquad (5\text{-}24)$$

where the subscript μ again denotes differentiation with respect to μ. Now from (5-20),

$$\Phi_\mu(\lambda,\mu) = \sum_{i=0}^{n-1} \frac{\partial\phi_i}{\partial\mu}\lambda^{n-i-1} \qquad (5\text{-}25)$$

and the quantities $\partial\phi_i/\partial\mu$ may be computed from the recursive relationship

$$\left.\begin{aligned}\frac{\partial\phi_i}{\partial\mu} &= \frac{\partial\phi_{i-1}}{\partial\mu}\bar{A}'(\mu) + \phi_{i-1}\bar{A}'_\mu(\mu) - \frac{1}{i}\,\mathrm{tr}\left[\frac{\partial\phi_{i-1}}{\partial\mu}\bar{A}'(\mu) + \phi_{i-1}\bar{A}'_\mu(\mu)\right], \\ &\qquad\qquad i = 1,2,\ldots,n, \\ \frac{\partial\phi_o}{\partial\mu} &= 0,\end{aligned}\right\} \quad (5\text{-}26)$$

that can be derived by differentiating the Faddeev relations (5-17). The matrix N can then be constructed by evaluating the quantities in (5-23) and (5-24) at the nominal value $\mu = \mu_o$ for $\lambda = \lambda_i^d$, $i = 1,2,\ldots,1$, with the aid of equations (5-25) and (5-26).

5.2.4.4 Solution algorithm

The following algorithm is now proposed for the evaluation of the optimal F.

Step (i): Choose an arbitrary starting value $F = F_o$. Set the penalty weight γ to some trial value, say $\gamma = 1$.

Step (ii): For the current value of F, construct the vector h and matrix N as outlined in §5.2.4.2 and §5.2.4.3.

Step (iii): Compute the increment δF according to Marquardt's modification of the formula (5-15). If δF is in some sense smaller than a prescribed tolerance, go to step (iv). Otherwise, increment F by δF and return to step (ii).

Step (iv): Check if pole placement has been accomplished to the desired tolerance either by solving for the closed-loop poles or by using a criterion such as $J_c \gtrless \varepsilon$. If the desired tolerance has been achieved, then a locally optimal value of F has been found. Otherwise, increase the penalty weight γ by a factor of, say, 10 and return to step (ii).

Remark

(R1). At step (iii), the increment δF may alternatively be determined in accordance with the Davidon-Fletcher-Powell, or similar, algorithm. For this purpose, expressions are required for J and $\partial J/\partial F$, and these are given by

$$J = h'h$$

$$\frac{\partial J}{\partial F} = 2Nh.$$

The problem in hand may conceivably have more than one local optimum. It is therefore advisable to repeat the computation starting from several different F_o in order to obtain the global optimum.

5.2.5 Numerical example

The technique developed above will now be illustrated by means of a numerical example. A third-order, two-input, two-output plant described by the equations:

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1/\mu & 1/\mu \\ 0 & 0 & -5 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1/\mu & 0 \\ 0 & 2-1/\mu \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x \end{aligned} \qquad (5\text{-}27)$$

is considered, the nominal value of the parameter μ being 1.0. The desired poles are -2, and $-1 \pm j2$. The problem is to find the feedback matrix F that gives exact pole placement for the nominal value of μ, while minimizing the sensitivity cost (5-9) with $\overline{S}_1 = \overline{S}_2 = 1$.

Here the feedback matrix F has four elements, which is one more than the number of poles that have to be assigned. Hence it is possible that there exists more than one F that gives exact pole placement for the nominal system. To test this possibility, the pole placement cost J_c given by (5-5) was minimized using Marquardt's algorithm for various starting F_o. It was found that every F_o tried converged to a different F with a vanishingly small value of J_c at the minimum, thus confirming that many, and possibly an infinite number of F, give exact pole placement. Two sample results are given in Table I where the corresponding values of the sensitivity cost function J_s are also tabulated. Thus it may be concluded that the problem stated above is well posed. A solution was attempted using the algorithm of §5.2.4.4, and it was found that all starting F_o tried converged to the values

$$F = \begin{bmatrix} 1.310 & 2.000 \\ 16.55 & -15.31 \end{bmatrix}$$

$$J_s = 43.103$$

with

$$J_c < 10^{-8}.$$

It is seen that the optimal F has a considerably smaller sensitivity to small parameter perturbations than either of the two values of F listed in Table I.

Starting F	Final F for exact pole placement	Sensitivity cost J_s
$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} -2.366 & 1.999 \\ 1.849 & -11.910 \end{bmatrix}$	238.100
$\begin{bmatrix} 10 & 10 \\ 10 & 10 \end{bmatrix}$	$\begin{bmatrix} -4.226 & 1.999 \\ 11.100 & -9.759 \end{bmatrix}$	485.985

Table I

5.3 DESIGN TECHNIQUE FOR LARGE PARAMETER VARIATIONS - PROBLEM B

In certain practical situations it may be unrealistic to assume that the perturbations in μ are small compared with its nominal value. If this is so, the design obtained using the previous formulation would not necessarily be the best possible. In what follows, the problem is reformulated so as to achieve a design where, over a prescribed range of μ, the closed-loop poles remain close to the desired locations.

5.3.1 Formulation of Problem B

Once again, the plant considered is described by the equations (5-1). The exact value of the parameter μ is now not known, but it is assumed to be distributed randomly in the interval

$$\mu_{min} \leqslant \mu \leqslant \mu_{max} \tag{5-28}$$

with a known probability density function $P(\mu)$.

Again, a set of desired poles (assumed distinct)

λ_i^d, $i = 1,2,\ldots,n$, is specified. Then an obvious generalization of the pole placement cost function (5-5) that allows for the variations in μ is

$$\hat{J}_c = \sum_{i=1}^{l} \int_{\mu_{min}}^{\mu_{max}} |\Delta(\lambda_i^d,\mu)|^2 P(\mu)d\mu . \qquad (5\text{-}29)$$

The problem therefore is to find the control law of the form (5-2) that minimizes the cost function (5-29).

Clearly it is impossible to achieve exact pole placement over a finite range of values of μ. Hence the optimal value of $\hat{J}_c$ will always be non-zero. However, although the optimal F may not give exact pole placement at even a single value of μ in the interval (5-28), it will be such that the closed-loop poles are in some sense as close as possible to the desired locations as μ varies in the interval (5-28).

Notice that unlike Problem A, Problem B is an unconstrained minimization problem. It is easy to show that the cost function (5-29) is a continuous function of the elements of F providing $A(\mu)$, $B(\mu)$, $C(\mu)$ and $P(\mu)$ are integrable functions. Also, $\hat{J}_c$ is clearly bounded below. Hence a solution to Problem B exists under very mild conditions.

Actually, the two problem formulations are not entirely unrelated. It can be shown that the solution of Problem B tends to that of Problem A (with the weights $\overline{S}_i = 1$), when it exists, as the parameter range $\mu_{max} - \mu_{min} \to 0$. Of course, Problem B becomes ill-conditioned computationally for this limiting condition.

5.3.2 Solution of Problem B

In practice it will usually be difficult, if not impossible, to explicitly perform the integration in (5-29). Hence a numerical integration technique, such as Simpson's rule, is used to approximate the integral by a finite sum giving

$$\hat{J}_c = \sum_{i=1}^{l} \sum_{j=1}^{q} \gamma_j |\Delta(\lambda_i^d, \mu_j)|^2 P(\mu_j) \qquad (5\text{-}30)$$

where the γ_j are constants with values depending on the integration rule being employed. The choice of the number of integration points q depends on the accuracy desired and the computational complexity that can be tolerated. Actually, since the performance index (5-29) is arbitrary, it need not be evaluated to any great accuracy. In fact one may even consider the finite sum (5-30) to be the performance index. Hence good designs may be obtained even when q is a relatively small number.

Observe that the cost function $\hat{J}_c$ is also of the sum-of-squares form. Clearly the components of $\hat{J}_c$ and their gradients with respect to F may be computed using the procedures outlined in §5.2.4.2 and §5.2.4.3. Hence there is no difficulty in performing the minimization of $\hat{J}_c$ using Marquardt's algorithm. Of course, any other gradient-type unconstrained minimization technique may be employed instead.

5.3.3 Illustrative example

The techniques just developed will be demonstrated on the plant (5-27) used for the previous example. The desired poles are again taken to be -2 and -1 ± j2. Now the parameter μ is assumed to be uniformly distributed in the interval

$$0.6 \leqslant \mu \leqslant 1.6,$$

i.e.,

$$P(\mu) = 1.0, \quad 0.6 \leqslant \mu \leqslant 1.6$$

Simpson's rule was used for integration with $q = 3$. The minimization was performed using Marquardt's algorithm, and it was found that all starting values of F tried produced the same solution:

$$F = \begin{bmatrix} -0.709 & 0.661 \\ -12.82 & -11.78 \end{bmatrix}$$

It is interesting to compare how the closed-loop poles vary with μ for the two different values of F obtained from the two different problem formulations. The results are summarised in Table II. For added interest, results are also shown for the first of the values of F given in Table I, which was obtained by satisfying the pole placement requirement only without considering sensitivity to parameter changes.

As would be expected, the optimal F for Problem A is more satisfactory from the pole placement standpoint for values of μ in the neighbourhood of the nominal value 1.0. However, it is less satisfactory for large deviations of μ from the nominal value and in fact results in instability when μ is less than about 0.6. The optimal F for Problem B though not giving exact pole placement for any value of μ nevertheless maintains the poles close to their desired values over a wide range of values of μ. Finally, it is

μ	Closed-loop poles		
	Optimal F for Problem B	Optimal F for Problem A	$F = \begin{bmatrix} -2.366 & 1.999 \\ 1.849 & -11.910 \end{bmatrix}$
0.5	-5.000 -0.340 ± j1.143	-5.000 -0.903, 2.903	-5.000 1.000 ± j1.932
0.6	-3.656 -0.956 ± j1.629	-3.749 -0.503, 0.919	-3.991 0.329 ± j2.139
0.8	-2.430 -1.497 ± j2.128	-2.325 -0.713 ± j1.626	-2.722 -0.514 ± j2.129
1.0	-2.266 -1.536 ± j2.205	-2.000 -1.000 ± j2.000	-2.000 -1.000 ± j2.000
1.2	-2.350 -1.467 ± j2.100	-2.009 -1.079 ± j2.032	-1.625 -1.270 ± j1.830
1.4	-2.523 -1.359 ± j1.954	-2.121 -1.083 ± j1.946	-1.443 -1.422 ± j1.630
1.6	-2.731 -1.241 ± j1.818	-2.283 -1.046 ± j1.828	-1.384 -1.495 ± j1.398
1.8	-2.936 -1.126 ± j1.705	-2.469 -0.988 ± j1.712	-1.235 -1.604 ± j1.305

Table II

evident that a typical F matrix chosen from only the pole placement standpoint would be markedly inferior to the optimal F for either Problem A or Problem B from the standpoint of sensitivity to parameter changes.

5.4 CONCLUSIONS

The problem of achieving pole placement in multivariable control systems with minimum sensitivity to plant parameter variations has been considered. Two design approaches have been developed. In one approach, exact pole placement is obtained at the nominal values of the plant parameters while minimizing a measure of the sensitivity of the poles to small parameter variations. This approach can be adopted when there are more feedback parameters than the minimum required for exact pole placement. In the second approach, the plant parameters are permitted to vary over a range that is large compared to their nominal values and the feedback gains obtained give approximate pole placement in a minimum mean square error sense over the prescribed range of plant parameter variations. The usefulness of the techniques developed has been illustrated by means of specific examples.

REFERENCES

[1] Sirisena, H.R. and Choi, S.S., "Optimal pole placement in linear multivariable systems using dynamic output feedback," Int. J. Contr., vol. 21, pp.661-671, April 1975.

[2] Morgan, B.S., Jr., "Sensitivity analysis and synthesis of multivariable systems," IEEE Trans. Automat. Contr., vol. AC-11, pp.506-512, July 1966.

[3] Reddy, D.C., "Eigenfunction sensitivity and parameter variation problem," Int. J. Contr., vol. 9, pp.561-568, 1969.

[4] Tzafestas, S.G. and Paraskevopoulous, P.N., "Sensitivity reduction in modal control systems," Franklin Inst. J., vol. 298, pp.29-43, July 1974.

[5] Rosen, J.B., "The gradient projection method for nonlinear programming, II," SIAM J., vol. 9, 1961.

[6] Fletcher, R. and Powell, M.J.D., "A rapidly convergent descent method for minimization," British Computer J., pp.163-168, June 1963.

[7] Marquardt, D.W., "An algorithm for least squares estimation of nonlinear parameters," SIAM J., vol. 11, pp.431-441, 1963.

[8] Gantmacher, F.R., The Theory of Matrices, vol. I. New York: Chelsea, 1959.

[9] Faddeev, D.K. and Faddeeva, V.N., Computational Methods of Linear Algebra. San Francisco: Freeman and Co., 1963.

[10] Sirisena, H.R. and Choi, S.S., "Pole placement in output feedback control systems for minimum sensitivity to plant parameter variations," Int. J. Contr., vol. 22, pp.129-140, July 1975.

CHAPTER 6

DESIGN OF SERVOMECHANISMS SUBJECT TO CONSTANT OR TIME-VARYING RANDOM DISTURBANCES

6.1 INTRODUCTION

In this chapter, the pole placement technique of Chapter 4 is extended to the design of dynamic compensators for servomechanisms subject to constant reference and disturbance inputs. The performance of the overall control system is assessed, not only in terms of its transient response characteristics but also on the magnitude of the steady-state error that may exist between the plant reference inputs and outputs.

Johnston [1] has established that if the constant input signals are unmeasurable, then system stability and output regulation (i.e., in the context of previous chapters, $x(t)$, $y(t) \to 0$ for $t \to \infty$) can be achieved through the use of an integral-state feedback law. His results have since been extended by other researchers. Thus, Davison and Smith [2] have derived the necessary and sufficient conditions for the existence of such an integral-state feedback control law. Similar results have also been obtained by Young and Willems [4] in the design of 'type-one' servomechanisms.

The case of measurable constant disturbances has been considered in [3] where it has been shown that steady-state output regulation can be achieved using a feedforward controller. The necessary and sufficient conditions under which the feedforward controller exists have also been derived.

The techniques described in this chapter have several features which are different from those contained in the abovementioned references. In §6.2, attention is first focussed on the design of dynamic compensators for servo-mechanisms where the constant inputs are unmeasurable. Some statistical properties of the input signals will be assumed. The order of the compensator is fixed a priori but is usually lower than that given in [2], [4]. The design trade-off is between the attainment of small (but not necessarily zero) steady-state error and the order of the compensator. Furthermore, this relaxation on the steady-state performance could result in the appearance of extra degrees of design freedom which may be used beneficially to improve system transient characteristics through suitable placement of the closed-loop poles. For this purpose, the unrestricted rank pole placement technique described in Chapter 4 is adopted because, unlike the unity-rank technique used in [4], greater flexibility in positioning the closed-loop poles may be expected. (See also Chapter 4.) In §6.2.6, the design of minimal-order dynamic compensators for simultaneous exact pole placement and zero steady-state error is considered.

The case where the constant inputs are assumed to be measurable is considered in §6.3. The results of [3] are specialized to show how a feedforward controller can be designed so that the steady-state error between the constant inputs and the plant output can be made exactly zero. The conditions governing the existence of such a controller are also derived.

By combining the results of §6.2 and §6.3, a two-step design procedure for the construction of a compensator for systems subject to measurable and unmeasurable constant inputs is presented in §6.4.

Attention is then directed towards the design of dynamic compensators for linear multivariable servomechanisms subject to random time-varying reference inputs, plant disturbances and measurement noise. Weston and Bongiorno [7] have determined the optimal compensator for this problem; however, the dimensionality of the optimal compensator may often be too high for realization purposes. A more practical solution could perhaps be obtained by constraining the compensator to be a dynamic system of low order, and this is the subject of §6.5. A quadratic performance index is employed and the resulting parameter optimization problem is treated using a state-variable formulation. It is shown that the servomechanism problem being considered can be reduced to a stationary stochastic regulator problem of the form considered in Chapter 3. Thus the direct-type solution algorithm proposed in §2.4.2 can be employed to determine the optimal compensator parameters. An illustrative numerical example is presented. Some of the results described in this section can also be found in a paper by Sirisena and Choi [5].

6.2 SERVOMECHANISM WITH UNMEASURABLE CONSTANT INPUTS

6.2.1 The system

The servomechanism considered is shown in Fig. 6.1 where the linear plant is given by

$$\dot{x}(t) = Ax(t) + Bu(t) + E_u w_u \tag{6-1}$$

$x(t) \varepsilon R^n$, $u(t) \varepsilon R^m$ are the plant state and control vectors respectively. $y(t) \varepsilon R^r$ denotes the plant output vector. A, B, E_u are constant matrices having appropriate dimensions. $w_u \varepsilon R^l$ is an external unmeasurable constant input and the error, $e(t)$, between w_u and $y(t)$ is given by

$$e(t) = Cx(t) + F_u w_u \tag{6-2}$$

The dynamic compensator is of the form

$$\left.\begin{aligned} \dot{z}(t) &= Pz(t) + Ne(t) \\ u(t) &= Hz(t) + Ge(t) \end{aligned}\right\} \quad (6\text{-}3)$$

where the order p of the compensator is fixed a priori. (P,N,H,G) are unknown matrices that remain to be determined. Recall that in forcing $e_{ss}(t) = 0$, where the subscript ss denotes the steady-state value, the compensator considered in [2], [4] is of order r with $P \equiv 0_{r,r}$, i.e., pure integration is used. Therefore, in using the present design technique for steady-state error control, the upper bound on the compensator order is $p = r$.

The unmeasurable constant input w_u will now be examined in greater detail. Suppose from time to time, w_u is changed from one value to another and the dynamics of the servomechanism shown in Fig. 6.1 are such that the time

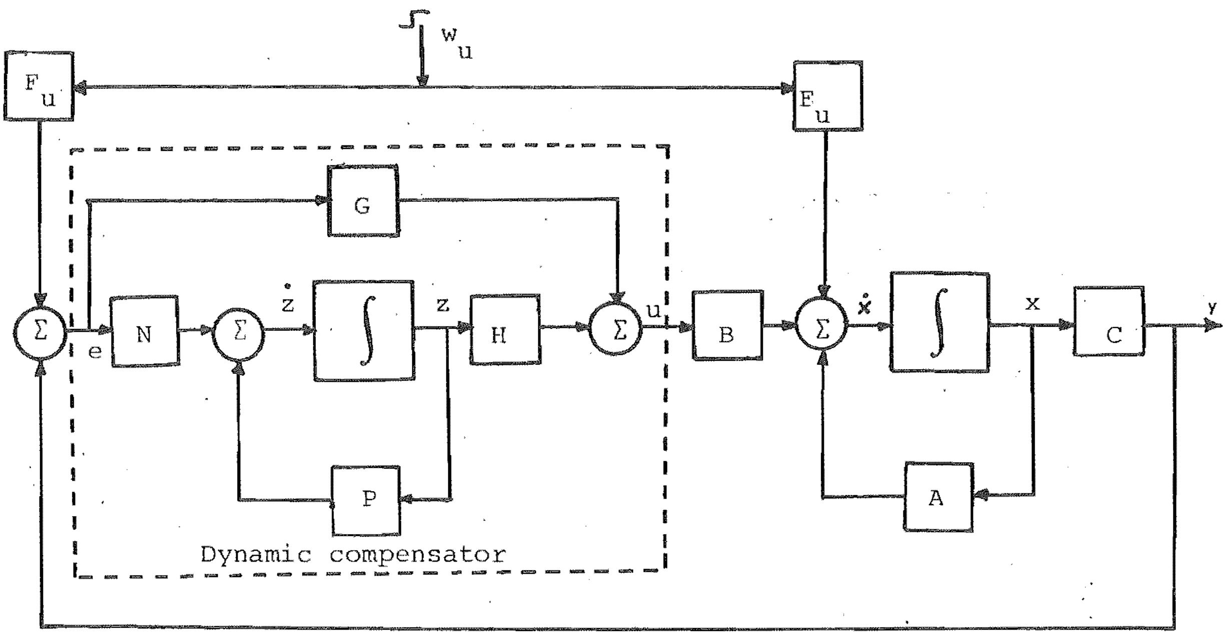

Fig. 6.1 Linear multivariable servomechanism with unmeasurable constant input w_u

interval during which w_u remains at a particular value is long compared to the time it takes $y(t)$ to essentially reach its steady-state value. Also, it is further assumed that the first- and second-order statistics describing these random variations of w_u are

$$E\{w_u\} = 0, \qquad E\{w_u w_u'\} = W_u \tag{6-4}$$

where W_u is a known 1×1 matrix. This is quite a common situation in practice [6] although the zero-mean assumption is made only for the convenience of analysis. A non-zero mean value may be regarded as a measurable disturbance, and can be handled by the technique of §6.3.

Equations (6-1), (6-2) and (6-3) can be combined to form the composite system

$$\dot{\bar{x}} \triangleq \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = (\bar{A} + \bar{B}K\bar{C})\bar{x} + (\bar{B}K\hat{F} + \bar{E})w_u \tag{6-5}$$

where

$$\left.\begin{array}{l} \bar{A} \triangleq \begin{bmatrix} A & 0_{n,p} \\ 0_{p,n} & 0_{p,p} \end{bmatrix}, \quad \bar{B} \triangleq \begin{bmatrix} B & 0_{n,p} \\ 0_{p,m} & I_p \end{bmatrix}, \quad \bar{C} \triangleq \begin{bmatrix} C & 0_{r,p} \\ 0_{p,n} & I_p \end{bmatrix} \\ K \triangleq \begin{bmatrix} G & H \\ N & P \end{bmatrix}, \quad \hat{F} \triangleq \begin{bmatrix} F_u \\ 0_{p,1} \end{bmatrix}, \quad \bar{E} \triangleq \begin{bmatrix} E_u \\ 0_{p,1} \end{bmatrix} \end{array}\right\} \tag{6-6}$$

The performance of the closed-loop system (6-5) will now be examined in terms of its steady-state and transient response characteristics.

6.2.2 Steady-state error measure

If the closed-loop system (6-5) is stable, then under steady-state condition, $\dot{x}(t), \dot{z}(t) \to 0$ as $t \to \infty$. Therefore, from (6-5),

$$\bar{x}_{ss} = -(\bar{A} + \bar{B}K\bar{C})^{-1}(\bar{B}K\hat{F} + \bar{E})w_u \qquad (6\text{-}7)$$

Stability of (6-5) implies that all the eigenvalues of $\bar{A} + \bar{B}K\bar{C}$ are situated on the LHP, and therefore $(\bar{A} + \bar{B}K\bar{C})^{-1}$ exists. Hence, from (6-2)

$$e_{ss} = [F_u - \check{C}(\bar{A} + \bar{B}K\bar{C})^{-1}(\bar{B}K\hat{F} + \bar{E})]w_u \qquad (6\text{-}8)$$

where $\check{C} \triangleq [C \quad 0]$. Now, to obtain acceptable steady-state performance, it is necessary to make e_{ss} as small as possible. A suitable performance index is therefore the mean-square error measure

$$J_{ss} = \tfrac{1}{2} E\{e'_{ss}\, e_{ss}\}. \qquad (6\text{-}9)$$

J_{ss} is positive definite and vanishes if and only if $e_{ss} \equiv 0$.

Using (6-4), J_{ss} can be evaluated to give

$$J_{ss} = \tfrac{1}{2}\,\mathrm{tr}\{[F_u - \check{C}(\bar{A}+\bar{B}K\bar{C})^{-1}(\bar{B}K\hat{F}+\bar{E})]'[F_u - \check{C}(\bar{A}+\bar{B}K\bar{C})^{-1}(\bar{B}K\hat{F}+\bar{E})]W_u\} \qquad (6\text{-}10)$$

For any given K, J_{ss} can thus be easily computed.

6.2.3 System transient performance

In this section, the transient response characteristics of (6-5) are studied in terms of the location of the closed-loop poles. Furthermore, stability of (6-5) must also be assured if the analysis shown in the last section is to be valid. Thus, it is necessary to find a suitable K such that the closed-loop eigenvalues, $\lambda_i(\bar{A} + \bar{B}K\bar{C})$ all i, are placed at or close to certain prescribed values. As can be seen from Chapter 4, there are several ways of achieving this. However, the unrestricted-rank pole placement technique described in §4.2 is to be preferred for the reasons stated there.

Retaining the notation used in Chapter 4, but denoting the pole placement cost by J_{ts} and replacing F (of Chapter 4) by K, the least-square pole placement cost is given by

$$J_{ts} = \tfrac{1}{2} \sum_{i=1}^{l} S_i |\Delta(\lambda_i^d)|^2 \qquad (6\text{-}11)$$

where $S_i (\geqslant 0)$ are constant weighting variables and $\Delta(\lambda)$ is the characteristic polynomial of the closed-loop matrix $\overline{A} + \overline{B}F\overline{C}$, i.e.,

$$\Delta(\lambda) \equiv |\lambda I_{n+p} - \overline{A} - \overline{B}K\overline{C}| = \lambda^{n+p} + a_1\lambda^{n+p-1} + \ldots + a_{n+p} \qquad (6\text{-}12)$$

For a given K and using the Faddeev algorithm [8], the coefficients of the characteristic polynomial a_i in (6-12) can be easily obtained. J_{ts} can be computed readily by repeated substitution of λ_i^d, all i, into (6-12). Clearly, $J_{ts} \equiv 0$ if and only if $\lambda_i \equiv \lambda_i^d$, all i, in which case exact pole placement is said to have occurred.

Finally, the steady-state and transient response performance indices [(6-10) and (6-11)] may now be combined. The servomechanism design problem may now be stated as follows:

<u>Problem Statement (S1)</u>

Given p and (6-5), find the parameter matrix K such that the joint performance index

$$J = J_{ts} + \gamma J_{ss} \qquad (6\text{-}13)$$

is minimized. $\gamma (\geqslant 0)$ is an arbitrary weighting constant appropriately chosen. This will be discussed later in §6.2.5.

Remark

(R1). It is clear that the design requirement for zero steady-state error has been relaxed in the present formulation (S1). Therefore it may be possible to construct compensators of order lower than that used in [2], [4] while still keeping the steady-state performance within design tolerances.

6.2.4 Analysis

Any direct search optimization algorithm may be used to find the minimal J given in (6-13). However, in so far as solution accuracy and computation efficiency are concerned, a gradient-type minimization technique is to be preferred. It is therefore necessary to derive a computable expression for $\partial J/\partial K$ and this is done in the next two sections.

6.2.4.1 Evaluation of $\partial J_{ss}/\partial K$

To facilitate the derivation that follows, first denote $A_o \triangleq (\bar{A} + \bar{B}K\bar{C})^{-1}$ and $B_o \triangleq \bar{B}K\hat{F} + \bar{E}$. Then introduce small perturbation in K, $\varepsilon\delta K$ (ε small) and expand (6-9) to first order in ε, using the relationship

$$(\bar{A} + \bar{B}K\bar{C} + \varepsilon\bar{B}\delta K\bar{C})^{-1} \approx A_o - \varepsilon A_o\bar{B}\delta K\bar{C}A_o$$

to obtain

$$e_{ss}(K + \varepsilon\delta K) \approx [F_u - \check{C}A_oB_o + \varepsilon\check{C}A_o\bar{B}\delta K(\bar{C}A_oB_o - \hat{F})]w_u$$

From (6-10), the increase in cost due to $\varepsilon\delta K$ is therefore given by

$$\delta J_{ss}(K) = \varepsilon \operatorname{tr}\{(\bar{C}A_oB_o-\hat{F})'\delta K'(\check{C}A_o\bar{B})'(F_u-\check{C}A_oB_o)W_u\} \tag{6-14}$$

Now, using a standard lemma due to Kleinman [10], the required gradient expression is given by

$$\frac{\partial J_{ss}}{\partial K} = (\check{C}A_o\bar{B})'(F_u-\check{C}A_oB_o)W_u(\bar{C}A_oB_o-\hat{F})' \tag{6-15}$$

Thus, for a given K, $\partial J_{ss}/\partial K$ is explicitly computable.

6.2.4.2 Evaluation of $\partial J_{ts}/\partial K$

The expression for $\partial J_{ts}/\partial K$ has already been obtained in Chapter 4 and is given by

$$\frac{\partial J_{ts}}{\partial K} = -\sum_{i=1}^{l} S_i\Delta^*(\lambda_i^d)\bar{B}'\Phi(\lambda_i^d)\bar{C}' \tag{6-16}$$

where $\Phi(\lambda)$ is the cofactor matrix of the determinant $|\lambda I - \bar{A} - \bar{B}K\bar{C}|$, and can be determined using Faddeev's algorithm [8].

Therefore, from (6-13), the complete gradient for J is given by

$$\frac{\partial J}{\partial K} = \frac{\partial J_{ts}}{\partial K} + \gamma\,\frac{\partial J_{ss}}{\partial K}, \qquad \gamma \geqslant 0 \tag{6-17}$$

and can be readily computed. Clearly, the expressions for costs ((6-10), (6-11)) and gradients ((6-15), (6-16)) enable the minimization of J to be performed using a gradient-type static optimization algorithm, such as that of DFP [9].

6.2.5 Possible design objectives

By suitably choosing the weighting constants S_i and γ, several design objectives may be attained. Some of these possibilities are:

(i) $\gamma = 1$, $S_i \to \infty$ all i. In effect, the steady-state error measure is minimized subject to exact pole-placement being attained. Clearly, a solution can exist only if the compensator order satisfies the lower bound for exact pole placement developed in Chapter 4, viz.

$$p \geqslant \frac{n - mr}{m + r - 1}$$

(ii) γ, S_i all i, are finite and non-vanishing. Then, the final design obtained is a compromise between exact pole placement and zero steady-state error. Large values of γ will cause the steady-state error to decrease but at the expense of the poles being shifted further away from the desired locations. This design objective will be illustrated by numerical examples in §6.2.7.

(iii) S_i is finite, all i, and $\gamma \to \infty$. Now the pole-placement cost is minimized subject to zero steady-state error being obtained. From the results of [4], zero steady-state error is obtainable only when pure integral control is used; hence a solution exists only if $p \geqslant r$. Actually, in this case it would be more efficient, computationally, to satisfy the zero steady-state error constraint at the outset by incorporating integral control, and then to concentrate on minimizing the pole-placement cost; this approach is pursued further in §6.2.6 below.

Remarks

(R2). For a given p, the design problems associated with (i) - (iii) above may have more than one local optimum, each with a different value of J. Therefore, if a gradient-type function minimization program is used to minimize J, it is necessary to try several different starting values of K so as to ensure that the global optimum has been reached.

(R3). In practice, the compensator order may have to be gradually increased from its minimum permissible value and the design procedure repeated until both the steady-state error and pole-placement error are within prescribed tolerances.

6.2.6 Design of dynamic compensator for simultaneous exact pole placement and zero steady-state error

To achieve simultaneous exact pole placement and zero steady-state error using output feedback, it is necessary that [2], [4]

(a) the linear plant (6-1) be controllable and observable, and

(b) $\rho \begin{bmatrix} A & B \\ C & O \end{bmatrix} = n + r.$

In this section, it will be assumed that both conditions are always satisfied. It is also known that [2], [4] the zero steady-state error constraint for constant disturbances can only be satisfied if at least a r^{th}-order pure integrator is used, i.e., if in the context of the compensator (6-3), a $r \times r$ submatrix of P, which corresponds to the pure integrator, is a null matrix. Consequently, it will be computationally unattractive to adopt, without any modifications, the design technique described in §6.2.4 since it is known beforehand that r^2 of these compensator parameters must be exactly zero.

It is for this reason that the dynamic compensator structure shown in Fig. 6.2 is proposed. The compensator incorporates a r^{th}-order pure integrator, thus ensuring that the zero steady-state error constraint is automatically satisfied. The remaining free parameters contained in (P_2, H_1, H_2, N_2, G) may therefore be used to achieve other design objectives, e.g. in the spirit of this chapter, the exact placement of all $n+p$ closed-loop poles. In this case, the design procedure only involves the minimization of J_{ts} (the least-square pole-placement cost) via an algorithm similar to that of Chapter 4.

The compensator of Fig. 6.2 can be described by equations of the form

$$\left.\begin{aligned} \dot{z}(t) &= Pz(t) + Ne(t) \\ u(t) &= Hz(t) + Ge(t) \\ \text{where} & \\ P \triangleq \begin{bmatrix} 0_{r,r} & 0_{r,p-r} \\ 0_{p-r,r} & P_2 \end{bmatrix}, & \quad H \triangleq \begin{bmatrix} H_1 & \vdots & H_2 \end{bmatrix}, \quad N \equiv N_2. \end{aligned}\right\} \quad (6\text{-}18)$$

Clearly, the order of this compensator satisfies

$$p \geqslant r \qquad \text{(B1)}$$

It is interesting to note that a second lower bound can also be derived in the following way. From §4.2.2.1, it is seen that there are, in effect, $(p-r)\times(m+r)+2mr$ independent parameters contained in (P_2, H_1, H_2, N_2, G). Exact placement of $n+p$ closed-loop poles implies that

$$n+p \leqslant (p-r)\times(m+r) + 2mr$$

i.e. a lower bound on p is given by

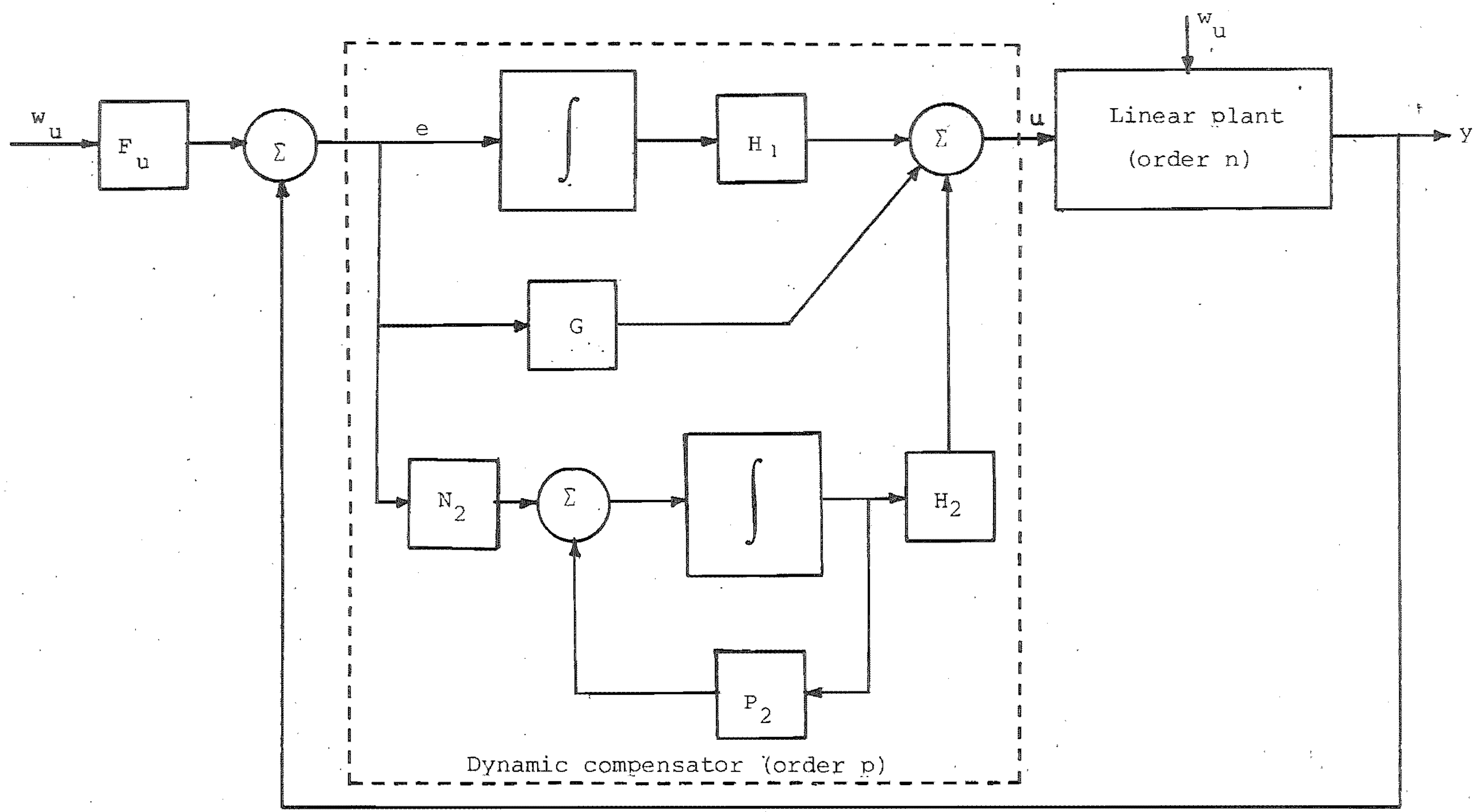

Fig. 6.2 Servomechanism configuration for achieving zero steady-state error and exact pole placement

$$p \geqslant \frac{n-mr+r^2}{m+r-1} \tag{B2}$$

where the RHS of (B2) is rounded up to an integer.

Thus, on combining (B1) and (B2) above, a new lower bound is obtained, i.e.

$$p' = \max\left[r, \frac{n-mr+r^2}{m+r-1}\right] \tag{6-19}$$

Finally, the minimal-order compensator for simultaneously achieving exact pole placement and zero steady-state error can be obtained via the following procedure:

Step (i): Set $p = p'$, given by (6-19).

Step (ii): Initialize search with guessed values for (P_2, H_1, H_2, N_2, G).

Step (iii): Minimize J_{ts} using the results of §4.2 in conjunction with a gradient-type optimization technique. If $J_{ts} < \varepsilon$ (a positive tolerance), then the required compensator has been obtained. Otherwise go to (iv).

Step (iv): Either (a) commence a new search with vastly different starting values for (P_2, H_1, H_2, N_2, G) and go to (iii),

or (b) increase p by 1 and go to (ii). Course (b) is followed only after repeated failures with course (a).

Remarks

(R4). The results of Brasch and Pearson [12], Kimura [13] can be extended to show that exact pole placement can be achieved if the order of the compensator (6-18) is sufficiently high, providing the plant (6-1) is observable and controllable. Hence, the above computational procedure will terminate for some finite compensator order p.

(R5). Young and Willems [4] and Davison and Smith [2] have also treated the problem of achieving zero steady-state error with exact pole placement. However, in addition to integral-of-error feedback, they also employ complete state feedback and this may be difficult to realise in practice.

(R6). Clearly, the assumption concerning the statistical properties of w_u made in §6.2.1 is unnecessary in view of the fact that $J_{ss} = 0$ for all unmeasurable constant inputs.

6.2.7 Numerical examples

An example from Young and Willems [4] is chosen to illustrate the design technique proposed in the earlier sections. The plant is

$$\mathring{x}(t) = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 1 & 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x(t)$$

The aim is to make y(t) follow a step input w_u where $W_u = I_2$. Clearly, in the context of §6.2.1, $F_u = -I_2$. The design procedure begins with the construction of a constant-gain controller , i.e. $p = 0$.

(a) Constant-gain controller

Suppose the closed-loop poles are to be placed at or in the vicinity of $-1 \pm j2$, -2. Let $S_i = 1$ for all i and

$\gamma = 1$. Using the results of §6.2.4 in conjunction with the Davidon-Fletcher-Powell (DFP) algorithm, all starting values of K tried result in the solution point

$$K = \begin{bmatrix} -1.4743 & -2.7640 \\ -1.3224 & -0.0226 \end{bmatrix}$$

with

$$J_{ss} = 0.11548$$

The closed-loop poles are situated at $-1.0005 \pm j2.0058$ and -2.0014. For this design, exact pole placement has virtually been achieved. Therefore, it may be possible to decrease J_{ss} further by increasing γ. Thus, let $\gamma = 10^4$ and $S_i = 1$ all i. The solution obtained is

$$K = \begin{bmatrix} -5.2938 & -4.3546 \\ -2.8098 & 0.8677 \end{bmatrix}$$

with

$$J_{ss} = 0.07224$$

The closed-loop poles are now located at $-1.5670 \pm j1.6528$, -7.8769. Thus it is seen that there is indeed a further reduction in the steady-state error but the poles are shifted further away from the desired locations.

(b) First-order dynamic compensator

Next, a first-order ($p = 1$) compensator is constructed. The desired poles are assumed to have values $-.5$, $-1 \pm j2$, -2. Let $S_i = 1$ for all i and $\gamma = 1$. The optimal solution is seen to be

$$K = \begin{bmatrix} -4.0711 & -1.7027 & 2.4076 \\ 2.8302 & 3.7830 & 2.8793 \\ -0.0957 & 0.0270 & -.0233 \end{bmatrix}$$

while

$$J_{ss} = 0.01838$$

Correspondingly, the poles are situated at $-.4999$, $-1.000 \pm j1.999$, -1.999 indicating that exact pole-placement has virtually been attained. Hence, it may be possible to reduce J_{ss} still further by increasing the penalty weight γ. Thus, when γ is increased to 10^4 while keeping S_i, all i, unchanged, the solution is

$$K = \begin{bmatrix} -4.9231 & -1.2611 & 3.7250 \\ 5.0993 & 4.3667 & 0.7847 \\ -0.0477 & 0.0633 & < 10^{-7} \end{bmatrix}$$

while the steady-state error has indeed been reduced further to

$$J_{ss} = 0.0004$$

The closed-loop poles are now situated at $-.4691$, $-1.994 \pm j1.996$, -1.9883,

Thus, in so far as steady-state error and pole positions are concerned, the last design is quite satisfactory. It is therefore not necessary to design a compensator of higher order although it is clear from §6.2.6 that for exact pole-placement and zero steady-state error, a compensator of at least order 2 must be constructed.

Finally, notice that in the last design, $P = k_{33} \approx 0$. This implies that the first-order dynamic compensator becomes very nearly like a pure integrator.

6.3 SERVOMECHANISM WITH MEASURABLE CONSTANT INPUTS

6.3.1 Problem formulation

When the constant inputs into the servomechanism are measurable, steady-state performance may be improved if a feedforward controller is employed. In the notation of §6.2.2, the plant is

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + E_m w_m \\ e(t) &= Cx(t) + F_m w_m \end{aligned}\right\} \quad (6\text{-}20)$$

where the constant input $w_m \varepsilon R^1$ in (6-20) is now assumed measurable.

The feedforward controller is given by

$$u = \bar{F}\, w_m \tag{6-21}$$

where $\bar{F}$ is an unknown $m \times 1$ constant matrix that remains to be determined such that $e(t) \to 0$ as $t \to \infty$. Now, from (6-20) and (6-21),

$$\dot{x}(t) = Ax(t) + (B\bar{F} + E_m)w_m \tag{6-22}$$

Providing A is stable, the system will settle down to a steady state given by

$$x_{ss} = -A^{-1}(B\bar{F} + E_m)w_m \tag{6-23}$$

Note that the stability of A implies the existence of A^{-1}. The steady-state error is thus given by

$$e_{ss} = [F_m - C\,A^{-1}(B\bar{F} + E_m)]w_m \tag{6-24}$$

The problem may now be stated:

Problem Statement (S2)

For a given system (6-2), find $\overline{F}$ such that e_{ss}, given by (6-24), vanishes for all constant inputs w_m.

6.3.2 Analysis

For any finite value of w_m, this implies, and is implied by, the condition that

$$F_m - C\,A^{-1}(B\overline{F} + E_m) \equiv 0_{r,1} \tag{6-25}$$

i.e.

$$(C\,A^{-1}B)\overline{F} = F_m - C\,A^{-1}\,E_m \tag{6-26}$$

From [11], equation (6-26) is satisfied exactly by the feedforward gain

$$\overline{F} = (C\,A^{-1}B)^{\dagger}\,(F_m - C\,A^{-1}E_m) \tag{6-27}$$

providing $\rho[C\,A^{-1}B] = r$. Moreover, when $\rho[C\,A^{-1}B] = r$,

$$(C\,A^{-1}B)^{\dagger} \equiv (C\,A^{-1}B)'[(C\,A^{-1}B)(C\,A^{-1}B)']^{-1} \tag{6-28}$$

Now it can be shown that

$$\rho[C\,A^{-1}B] = r \Longleftrightarrow \rho\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = n + r\,.$$

Hence, to summarize, feedforward control giving zero steady-state error exists and is given by (6-27), (6-28) providing

(a) A is stable

(b) $\rho\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = n + r$

6.4 A SERVOMECHANISM DESIGN PROCEDURE

The results of previous sections may now be combined and applied to the design of servomechanisms subject to both measurable and unmeasurable constant inputs. Essentially, the design procedure consists of two steps.

Step 1: First, disregard the measurable inputs w_m and consider the system design for only unmeasurable inputs w_u. Using the notation of §6.2.1, find K as defined in (6-6) such that

(i) the composite system is stable and $\lambda_i(\bar{A} + \bar{B}K\bar{C})$ are at or near to prescribed positions on the complex plane,

(ii) the steady-state error e_{ss} due to the unmeasurable constant inputs w_u is minimized.

These two design objectives can be attained using the techniques of §6.2.

Step 2: Now suppose the compensated system obtained from Step 1 contains measurable inputs w_m, i.e.

$$\left.\begin{aligned} \dot{\bar{x}} &= (\bar{A} + \bar{B}K\bar{C})\bar{x} + (\bar{B}K\hat{F} + \bar{E})w_u + \bar{B}\bar{u} + E_m w_m \\ \bar{e} &= \check{C}\,\bar{x} + F_m w_m \end{aligned}\right\} \qquad (6\text{-}29)$$

with $\check{C} \triangleq [C \quad 0]$. Find the control law

$$\bar{u} = \bar{F}_m w_m \qquad (6\text{-}30)$$

where $\bar{F}_m$ is the unknown feedforward gain matrix. In this case, the steady-state error $\bar{e}_{ss}$ contains two terms:

$$\bar{e}_{ss} = \check{C}(\bar{A}+\bar{B}K\bar{C})^{-1}(\bar{B}K\hat{F}+\bar{E})w_u + [F_m - \check{C}(\bar{A}+\bar{B}K\bar{C})^{-1}(\bar{B}\bar{F}_m + E_m)]w_m \qquad (6\text{-}31)$$

The first term on the RHS of (6-31) is the error due to the unmeasurable inputs w_u. The magnitude of this error will depend on the design obtained from Step 1. However, the second term on the RHS of (6-31) corresponds to the error due to the measurable inputs w_m which, as has already been discussed in §6.3, may be made zero provided the following conditions are satisfied:

(i) $\overline{A} + \overline{B}K\overline{C}$ is stable. This is guaranteed as a result of Step 1, and correspondingly, $(\overline{A} + \overline{B}K\overline{C})^{-1}$ exists.

(ii) Identify $(\check{C}, \overline{A} + \overline{B}K\overline{C}, \overline{B})$ in (6-29) with (C,A,B) in (6-20), then it is necessary that

$$\rho \begin{bmatrix} \overline{A} + \overline{B}K\overline{C} & \overline{B} \\ \check{C} & 0 \end{bmatrix} = n + r + p$$

It can be shown [3] that this condition is equivalent to

$$\rho \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = n + r$$

Also, the required feedforward gain is given by

$$\overline{F}_m = [\check{C}(\overline{A} + \overline{B}K\overline{C})^{-1}\overline{B}]^{\dagger}[F_m - \check{C}(\overline{A} + \overline{B}K\overline{C})^{-1}E_m] \qquad (6\text{-}32)$$

Since K is already determined from Step 1 and all the matrices appearing in (6-32) are known, the feedforward gain matrix is easily determined.

6.5 SERVOMECHANISM WITH STATIONARY RANDOM INPUTS

Some practical systems may be subjected to disturbances which are not constant but vary randomly with time. In such cases, the disturbances (and reference inputs) may usually be modelled as stationary Markov processes. The design of output feedback controllers for such systems is considered in this section.

6.5.1 The problem

The system under consideration is depicted in Fig. 6.3. The plant to be controlled is described by

$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}\right\} \qquad (6\text{-}33)$$

while the reference signal y_r, the plant disturbance y_d and the measurement noise y_m are the stationary Markov processes:

$$\left.\begin{aligned} \dot{x}_r &= A_r x_r + w_r \\ y_r &= C_r x_r \end{aligned}\right\} \qquad (6\text{-}34)$$

$$\left.\begin{aligned} \dot{x}_d &= A_d x_d + w_d \\ y_d &= C_d x_d \end{aligned}\right\} \qquad (6\text{-}35)$$

$$\left.\begin{aligned} \dot{x}_m &= A_m x_m + w_m \\ y_m &= C_m x_m \end{aligned}\right\} \qquad (6\text{-}36)$$

respectively, where w_r, w_d and w_m are white-noise processes with known variances.

The problem is to determine the fixed-order dynamic compensator of the form

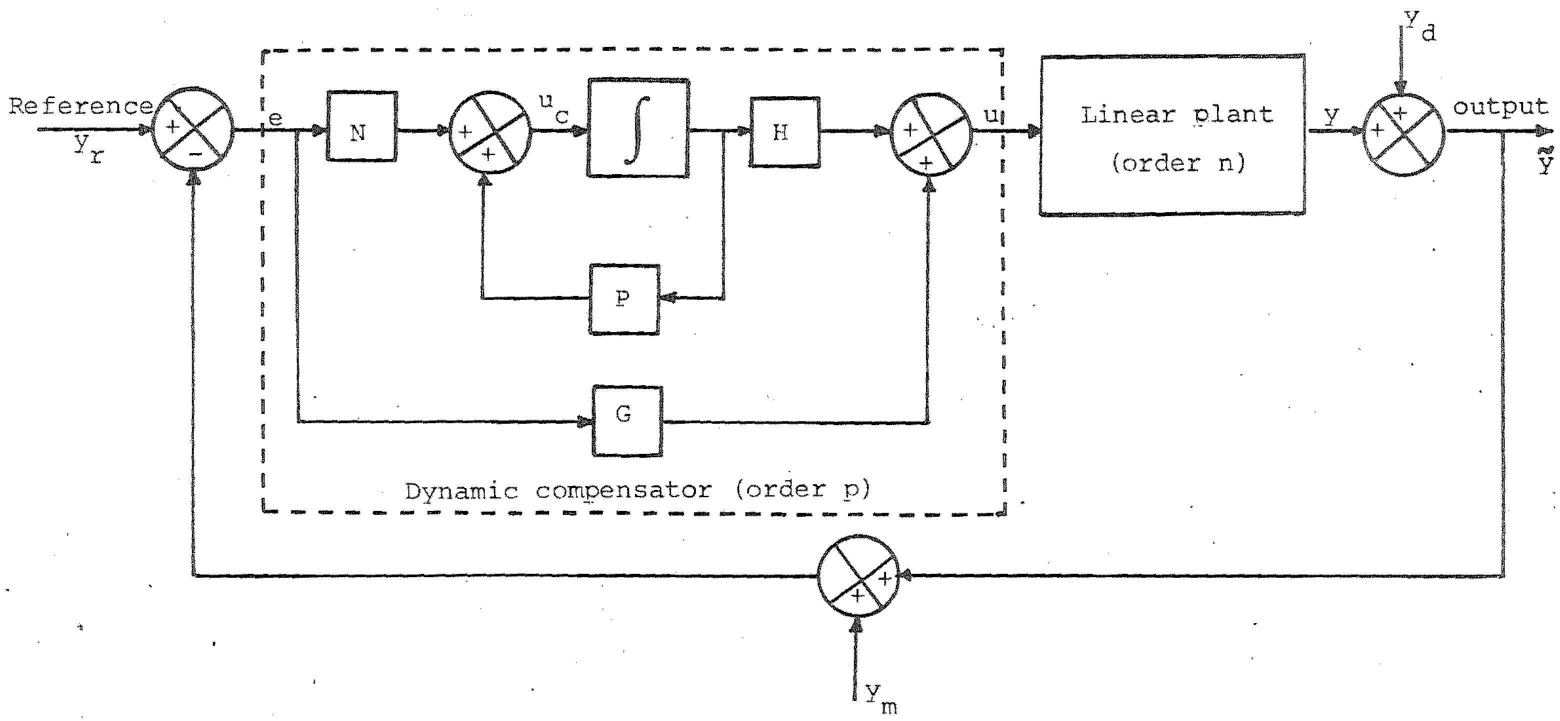

Fig. 6.3 Multivariable system configuration

$$\left.\begin{aligned} \dot{z} &= Pz + Ne \\ u &= Hz + Ge \end{aligned}\right\} \quad (6\text{-}37)$$

where e is the measured error signal, which minimizes the quadratic performance criterion

$$J = \tfrac{1}{2} E\{(y_r - \tilde{y})'Q(y_r - \tilde{y}) + u_c'R_c u_c + u'R_p u\} \quad (6\text{-}38)$$

under stationary conditions, where Q is positive semi-definite and R_c and R_p are both positive definite. Note that (6-38) weights the compensator input u_c in addition to the plant input u and the tracking error.

It is tacitly assumed that the order of the compensator (6-37) is sufficiently high for the system to be stabilizable. However, the compensator order need be no greater than that of the (unconstrained) optimal dynamic compensator which consists of a Bryson-Johansen filter of order $n + \dim(x_r) + \dim(x_d) + \dim(x_m) - \dim(y)$ followed by a memoryless linear transformation.

Finally, if so desired, the canonical forms (C1) and (C2) of §2.2.2 may be adopted for the matrices P and N respectively.

6.5.2 Analysis

The system under consideration can be described by the composite state equation

$$\dot{\bar{x}} = (\bar{A} + \bar{B}F\bar{C})\bar{x} + w \quad (6\text{-}39)$$

where

$$\bar{x} \triangleq \begin{bmatrix} x \\ z \\ x_r \\ x_d \\ x_m \end{bmatrix} ; \quad w \triangleq \begin{bmatrix} 0 \\ 0 \\ w_r \\ w_d \\ w_m \end{bmatrix} ; \quad \bar{A} \triangleq \begin{bmatrix} A & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_r & 0 & 0 \\ 0 & 0 & 0 & A_d & 0 \\ 0 & 0 & 0 & 0 & A_m \end{bmatrix} ; \quad \bar{B} \triangleq \begin{bmatrix} B & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$F \triangleq \begin{bmatrix} G & H \\ N & P \end{bmatrix} ; \quad \bar{C} \triangleq \begin{bmatrix} -C & 0 & C_r & -C_d & -C_m \\ 0 & I & 0 & 0 & 0 \end{bmatrix}$$

The problem then is to determine the matrix F of compensator parameters that minimizes the performance criterion (6-38).

Now the tracking error

$$y_r - \tilde{y} = [-C \quad 0 \quad C_r \quad -C_d \quad 0]\bar{x} = \hat{C}\bar{x} \tag{6-40}$$

while the plant and compensator inputs are given by

$$\begin{bmatrix} u \\ u_c \end{bmatrix} = F\bar{C}\bar{x} \tag{6-41}$$

Hence the cost

$$J = \tfrac{1}{2} E\{\bar{x}'\hat{C}'Q\hat{C}\bar{x} + \bar{x}'\bar{C}'F'RF\bar{C}\bar{x}\} \tag{6-42}$$

where

$$R \triangleq \begin{bmatrix} R_p & 0 \\ 0 & R_c \end{bmatrix} \tag{6-43}$$

i.e.

$$J = \tfrac{1}{2} \operatorname{tr}\{(\hat{C}'Q\hat{C} + \bar{C}'F'RF\bar{C})\bar{X}\} \tag{6-44}$$

where $\bar{X} \equiv E(\bar{x}\bar{x}')$. Notice that J is of a form similar to that obtained in §3.2 and hence, the following results are obtained. Assuming that the closed-loop system matrix

$\bar{A}_o = \bar{A} + \bar{B}F\bar{C}$ is stable,

$$\frac{\partial J}{\partial F} = RF\bar{C}X\bar{C}' + \bar{B}'K X\bar{C}' \tag{6-45}$$

where X and K are also the symmetric positive semi-definite solutions of the bilinear matrix equations

$$X\bar{A}_o' + \bar{A}_o X + W = 0 \tag{6-46}$$

$$K\bar{A}_o + \bar{A}_o'K + \hat{C}'Q\hat{C} + \bar{C}'F'RF\bar{C} = 0 \tag{6-47}$$

and

$$W = E\{w\,w'\}.$$

Since computable expressions are available for both the cost and its gradient with respect to the compensator parameters, the optimal values of these parameters may be obtained by employing a gradient search algorithm following the method of §2.4.2.

Remark

(R7). The sparsity of the matrix $\bar{A}_o$ can be exploited to greatly reduce the computation effort involved in solving (6-46) and (6-47). By appropriately partitioning the symmetric matrix K, (6-47) can be decomposed into ten bilinear matrix equations of much lower dimensionality which may then be solved in sequence. The covariance matrix X may be partitioned similarly; in this case the six submatrices corresponding to the states of the stochastic processes (6-34) - (6-36) are invariant and can be pre-computed. The solution of (6-46) therefore reduces to the solution of four lower-dimensional bilinear matrix equations.

6.5.3 Numerical example

The algorithm was employed to design a first-order compensator for the second-order plant

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 0]x$$

The reference signal, plant disturbance and measurement noise were respectively modelled by

$$\dot{y}_r = -5y_r + w_r, \qquad \mathrm{var}(w_r) = 5.0$$

$$\dot{y}_d = -y_d + w_d, \qquad \mathrm{var}(w_d) = 0.1$$

$$\dot{y}_m = -25y_m + w_m, \qquad \mathrm{var}(w_m) = 10.0$$

with w_r, w_d and w_m being uncorrelated. The weighting matrices chosen were

$$Q = 10, \qquad R_c = 0.001, \qquad R_p = 0.01$$

The redundant parameter N was fixed at 1. Then after five iterations of the Davidon-Fletcher-Powell algorithm, the remaining compensator parameters converged to

$$P = -2.94, \qquad H = -12.76, \qquad G = 5.32$$

the minimum cost being 2.487. Note that the optimal dynamic compensator for this problem is of order 4.

6.6 CONCLUSION

The control of LMS subject to both measurable and unmeasurable constant inputs has been considered in this chapter. For systems with unmeasurable constant inputs, the technique proposed in §6.2.4 results in a compensated system having both minimum mean-square steady-state error and a desired transient response characteristic through suitable positioning of the closed-loop poles. This technique is useful when zero steady-state error is not of prime importance since it enables satisfactory steady-state and transient performance to be obtained with relatively low order compensators. Next, the design of minimum order compensator for the purpose of achieving exact pole-placement and zero steady-state error was considered. It was shown that this design aim may be realised through the use of the compensator (6-18) in conjunction with the lower bound and computational procedure presented in §6.2.6. Unlike previous work in this area [2], [4], the use of complete state feedback is avoided.

For systems subject also to measurable constant disturbances, it was shown how the steady-state error component due to these disturbances could be zeroed by the use of feedforward control providing the two conditions shown in §6.3.2 are satisfied.

Consideration has also been given to the design of fixed-order time-invariant dynamic compensators for linear multivariable servomechanism subject to randomly time-varying disturbances. A quadratic performance criterion has been employed for which a computational procedure similar to that described in §2.2 has been proposed to obtain the optimal compensator parameters. Acceptable control of such servomechanism may thereby be attained without the need for optimal controllers incorporating high-dimensional state estimators.

REFERENCES

[1] Johnson, C.D., "Optimal control of the linear regulator with constant disturbances," IEEE Trans. Automat. Contr., vol. AC-13, pp.416-420, Aug. 1968.

[2] Davison, E.J. and Smith, H.W., "Pole assignment in linear time-invariant multivariable systems with constant disturbances," Automatica, vol. 7., pp.489-498, 1971.

[3] Smith, H.W. and Davison, E.J., "Design of industrial regulators: Integral feedback and feedforward control," Proc. I.E.E., vol. 119, pp.1210-1216, Aug. 1972.

[4] Young, P.C. and Willems, J.C., "An approach to the multivariable servomechanism problem," Int. J. Contr., vol. 15, pp.961-979, 1972.

[5] Sirisena, H.R. and Choi, S.S., "Design of optimal constrained dynamic compensators for linear stationary stochastic servomechanisms," Int. J. Contr., vol. 20, pp.363-368, 1974.

[6] Kwakernaak, H. and Sivan, R., Linear Optimal Control Systems. New York: Wiley-Interscience, 1972.

[7] Weston, J.E. and Bongiorno, J.J., Jr., "Extension of analytical design techniques to multivariable feedback control systems," IEEE Trans. Automat. Contr., vol. AC-17, pp.613-620, Oct. 1972.

[8] Gantmacher, F.R., Matrix Theory. New York: Chelsea, 1959.

[9] Fletcher, R. and Powell, M.J.D., "A rapidly convergent descent method for minimization," British Computer J., pp.163-168, June 1963.

[10] Kleinman, D.L., "On the linear regulator problem and the matrix Riccati equation," M.I.T. Electronic Systems Lab., Cambridge, Mass., Tech. Report. ESL-R-271, 1966.

[11] Penrose, R.A., "A generalized inverse for matrices," Proc. Cambridge Phil. Soc., vol. 51, pp.406-413, 1955.

[12] Brasch, F.M., Jr. and Pearson, J.B., "Pole placement using dynamic compensators," IEEE Trans. Automat. Contr., vol. AC-15, pp.34-43, Feb. 1970.

[13] Kimura, H., "Pole assignment by gain output feedback," IEEE Trans. Automat. Contr., vol. AC-20, pp.509-516, Aug. 1975.

CHAPTER 7

DESIGN OF MINIMAL-ORDER OBSERVER-COMPENSATORS FOR LINEAR MULTIVARIABLE SYSTEMS

7.1 INTRODUCTION

Based on recent theoretical developments in observer design, an alternative to the output feedback control techniques described in Chapters 2 and 4 is considered in this chapter. Initially, the Luenberger observer theory [12] was developed for the asymptotic estimation of the entire state of a linear noise-free multivariable system. The estimated state vector may then be used to implement a closed-loop state feedback law. The control scheme is depicted in Fig. 7.1. The Luenberger observer is of order $n-r$ [1] where n, r are the dimensions of the state and output vectors respectively. Furthermore, the observer poles may also be chosen arbitrarily so long as they do not coincide with the original plant poles $\lambda_i(A)$, $i=1,\ldots,n$.

Actually for control purposes, it is clearly only necessary to estimate a linear vector function of the states rather than the complete state vector, and recent research has shown that this may be achieved by means of *reduced-order observers*, as depicted in Fig. 7.2.

Thus, Luenberger [1] has shown that a scalar linear function of the state may always be estimated by an observer

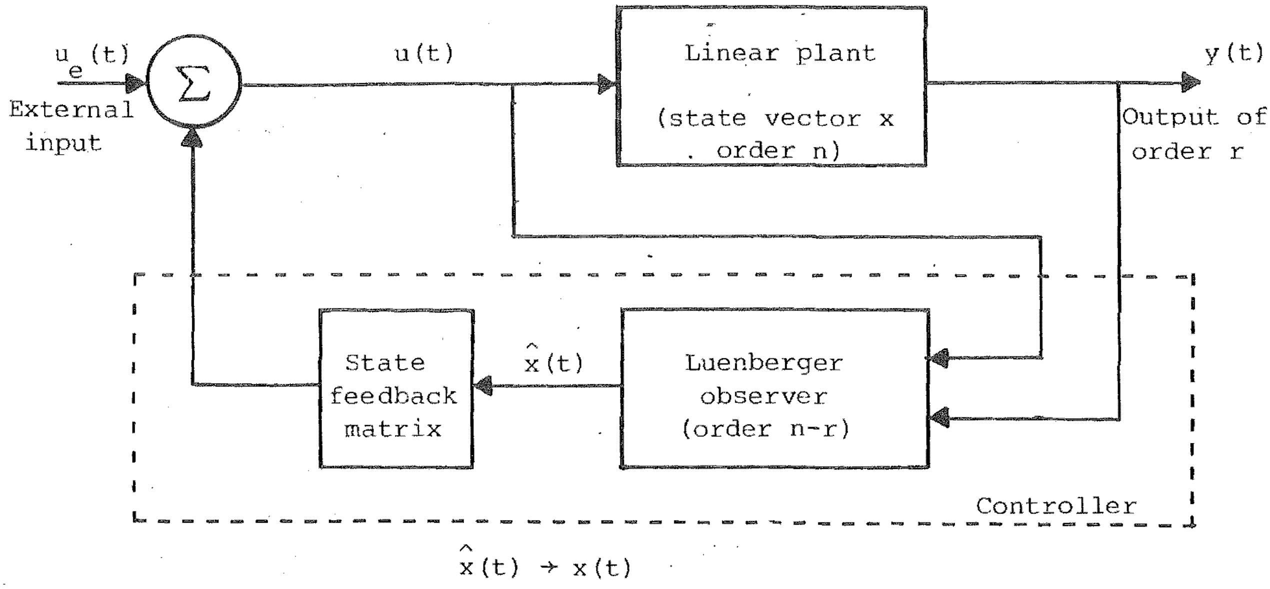

Fig. 7.1 Control of linear multivariable system using Luenberger observer

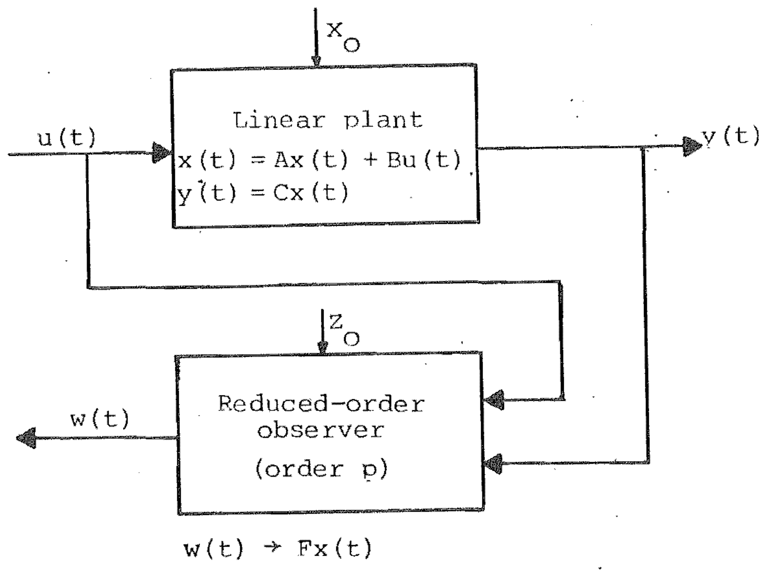

Fig. 7.2 Reduced-order observer to estimate linear function of state

of order $\nu_o - 1$ having arbitrary dynamics, where ν_o is the observability index of the system. Moreover, Fortmann and Williamson [2] have shown that the order of the observer required for this purpose may be reduced still further by foregoing the luxury of arbitrarily specifying its poles. These results have prompted the study of the more general problem of finding minimal-order observers for estimating vector functions of the state. Three distinct problems can be distinguished depending on whether the observed poles are unrestricted, restricted to lie in the LHP or specified arbitrarily at the outset, and henceforth these problems will be referred to as *the free-pole observer problem*, *the stable observer problem* and *the fixed-pole observer problem* respectively.

Luenberger [1] solved the fixed-pole observer problem but only for SIMO and MISO systems; however, it was only recently that Roman and Bullock [3] showed that his solutions were indeed of minimal-order. The free-pole observer problem was solved by Fortmann and Williamson [2], again only for SIMO and MISO systems; the proposed extension of their technique to MIMO systems cannot be guaranteed to yield minimal-order observers. Solutions to the stable observer problem may also be obtained using the technique of Fortmann and Williamson but not in a systematic (i.e., algorithmic) manner; instead stable observers must be sought among the solutions to the free-pole problem.

In the case of MIMO systems, the free-pole observer problem has been solved by Roman and Bullock [3] via realization theory. Solutions can also be obtained using their technique for the stable and fixed-pole observer problems, but again not in a systematic manner suitable for programming on a computer. However, their approach does provide considerable insight into the problem.

Anderson *et al.* [4] have proposed using decision methods for minimal-order observer design. However, the computational effort involved in decision methods increases exponentially with the number of parameters to be determined. Hence, despite the success of Moore and Ledwich [5] in reducing the number of free observer parameters for use with this technique to an absolute minimum, it does not appear that this approach will be a practical proposition.

In contrast to previous approaches, the present chapter describes a technique whereby the observer design problem is transformed into an optimization problem in the free observer parameters. Solutions are thereby easily obtained for both the free-pole and fixed-pole observer problems for MIMO systems. Also, by generalizing an idea contained in [5] and adopting a special form of the observer matrix that always ensures a prescribed degree of stability, solutions are easily obtained for the stable observer problem.

Suppose $Fx(t)$ in Fig. 7.2 is the linear optimal state feedback law that minimizes the usual quadratic performance index. When $w(t)$ is fed back directly to $u(t)$, a closed-loop system is obtained where the observer becomes a dynamic compensator. The performance of the compensated system is investigated in §7.5.1. The corresponding results for the special case $p = n - r$ (or Luenberger observer) have been obtained by Bongiorno and Youla [11].

Finally, instead of the minimization of the quadratic performance index, the suitability of using the observer-compensator to implement linear state feedback laws obtained for pole placement or decoupling is also discussed.

Some of the results obtained in this chapter also appear in a paper by Sirisena and Choi [14].

7.2 THE OBSERVER DESIGN PROBLEM

7.2.1 Problem statement

Consider an observable system

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_o \\ y(t) &= Cx(t) \end{aligned}\right\} \qquad (7\text{-}1)$$

$x(t) \varepsilon R^n$, $y(t) \varepsilon R^r$ and $u(t) \varepsilon R^m$. (A,B,C) are constant matrices of appropriate dimensions. C is of full rank. The aim is to design a p^{th}-order $(0 \leqslant p \leqslant n-r)$ observer of the form

$$\left.\begin{aligned} \dot{z}(t) &= Dz(t) + Ky(t) + Gu(t), \quad z(0) = z_o \\ w(t) &= Hz(t) + Ey(t) \end{aligned}\right\} \qquad (7\text{-}2)$$

such that

$$\lim_{t \to \infty} \|w(t) - Fx(t)\| \to 0 \qquad (7\text{-}3)$$

for all x_o, z_o, $u(\cdot)$ and a given $1 \times n$ matrix F. Fig. 7.3 shows the LMS.

The *stable observer problem* is thus to find a set of matrices (D,K,G,H,E) with D stable but otherwise unrestricted such that (7-3) is satisfied.

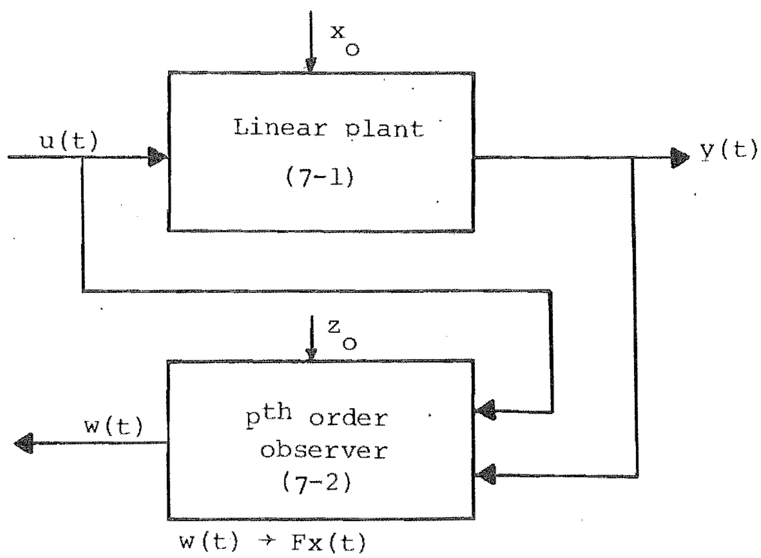

Fig. 7.3 Asymptotic estimate of Fx(t) using reduced-order observer

The *fixed-pole observer problem* is similarly defined, except that $\lambda_i(D)$, $i = 1,\ldots,p$ are specified.

There is no need to consider the *free-pole observer problem* because unstable "solutions" of this problem will not satisfy (7-3) while stable solutions of this problem will also be solutions of the stable observer problem.

7.2.2 Preliminary results

The necessary and sufficient conditions that guarantee the existence of an observer are well known, e.g. [2]. If there exists a constant full-rank matrix T such that $z(t) \to Tx(t)$ asymptotically, then these conditions are:

condition (i) : $TA - DT = KC$ (7-4)

condition (ii) : $HT + EC = F$ (7-5)

condition (iii) : $G = TB$ (7-6)

and condition (iv) : $\lambda_i(D)$, $i=1,\ldots,p$ are stable (7-7)

Conditions (i) - (iii) are non-linear algebraic equations in the 6-tuple (D,K,G,H,E,T) while (iv) represents p polynomial inequalities in these quantities. Thus, at first glance, the solution of the observer problem appears to be quite formidable. However, it is shown in the sequel that condition (iv) can be taken care of by adopting a special canonical form for D and that the remaining conditions can be reduced to an optimization problem in just the matrices (D,K).

Firstly, for a given (D,K), T can be obtained by solving (7-4) using e.g., the algorithm of [6]. T is also unique if and only if $\lambda_i(D) \neq \lambda_j(A)$, all i,j, [7]. Also, once T is known, G can be obtained immediately from (7-6). Thus, condition (iii) may be disregarded at this stage. In the next section, condition (ii) is examined in greater detail.

7.2.3 Geometric interpretation of condition (ii)

It is assumed, without loss of generality, that

$$C = \left[0_{r,n-r} \;\vdots\; I_r \right] \tag{7-8}$$

From (7-5) and (7-8) it is seen that

$$HT_1 = F_1 \tag{7-9}$$

$$HT_2 = E + F_2 \tag{7-10}$$

where $T \triangleq \left[T_1 \;\vdots\; T_2 \right]$, $F \triangleq \left[F_1 \;\vdots\; F_2 \right]$; $T_1 \varepsilon R^{p\times(n-r)}$, $F_1 \varepsilon R^{1\times(n-r)}$. Geometrically, (7-9) implies that all rows of F_1 must lie in the row space of T_1. When this occurs, the existence of H is assured. From (7-10), E can be easily computed if H is obtained from (7-9). Thus, E may be omitted from subsequent design considerations.

To simplify (7-9) still further, it will be convenient to introduce the concept of a projection matrix [8].

Definition

Consider a full-rank $q \times s$ matrix Φ, $s \geqslant q$. Let $\tilde{Q}$ denote the q-dimensional subspace spanned by the rows of Φ, and denote by Q its orthogonal complement subspace, of dimension s-q, such that $E_s = Q \oplus \tilde{Q}$.

From [8], the projection matrix P that takes any vector in E_s into $\tilde{Q}$ is given by

$$P = \Phi'(\Phi\Phi')^{-1}\Phi \tag{7-11}$$

Since Φ is of full rank, $(\Phi\Phi')^{-1}$ exists.

Similarly, the projection matrix $\tilde{P}$ that takes any vector in E_s into Q is given by

$$\tilde{P} = I_s - \Phi'(\Phi\Phi')^{-1}\Phi \tag{7-12}$$

Returning to the observer problem, identify T_1 in (7-9) with Φ in (7-12). For all rows of F_1 to lie in the row space of T_1, the projection of each of these rows into Q, the $(n-r-p)^{th}$-dimensional subspace orthogonal to that spanned by the rows of T_1, must identically be a zero vector, i.e.,

$$F_1\left[I_{n-r} - T_1'(T_1T_1')^{-1}T_1 \right] \equiv 0_{1,n-r} \tag{7-13}$$

It is assumed, without loss of generality, that T_1 is of full rank and so $(T_1T_1')^{-1}$ exists. If T_1 is not of full rank, it can usually be made so by slight perturbation of (D,K). Failing this, the linearly dependent rows of T_1 can be eliminated with a corresponding reduction in the observer order.

Furthermore, explicit expressions relating (H,E) with T_1 are obtained, as follows:

$$\left.\begin{aligned} H &= F_1T_1'(T_1T_1')^{-1} \\ E &= F_2 - F_1T_1'(T_1T_1')^{-1}T_2 \end{aligned}\right\} \tag{7-14}$$

In view of the foregoing results, the observer design problem essentially reduces to that of finding a pair (D,K) that satisfies (7-13). Note that T_1 appearing in (7-13) is an implicit function of (D,K) because of (7-4).

7.3 SOLUTION OF MINIMAL-ORDER STABLE OBSERVER PROBLEM

7.3.1 Canonical form for D

It can be verified easily that if (D,K,G,H,E,T) constitutes a solution to the observer problem, so does the 6-tuple (SDS^{-1}, SK, SG, HS^{-1}, E, ST) where S is any non-singular $p \times p$ matrix. Hence, by suitably choosing S, D can be transformed into convenient canonical forms.

Now, consider the following block-diagonal form of D

$$D = \begin{bmatrix} D_{11} & & & 0 \\ & D_{22} & & \\ & & \ddots & \\ 0 & & & D_{ss} \end{bmatrix} \qquad \text{(C1)}$$

where the diagonal blocks are 2×2 matrices of the form

$$D_{ii} = \begin{bmatrix} 0 & -\alpha_{i1}{}^2\varepsilon-\alpha_{i2}{}^2 - \varepsilon^2 \\ 1 & -\alpha_{i1}{}^2 - 2\varepsilon \end{bmatrix}, \quad i = \begin{cases} 1,\ldots,p/2, & p \text{ even} \\ 1,\ldots,(p-1)/2, & p \text{ odd} \end{cases}$$

except that when the observer order is odd, there is an additional 1×1 block of the form

$$D_{ii} = -\alpha_{i1}{}^2 - \varepsilon, \quad i = (p+1)/2, \; p \text{ odd.}$$

and where ε is an arbitrary positive constant. It can be easily shown that $\mathrm{Re}\{\lambda_i(D)\} \leqslant -\varepsilon$, all i. Thus, any prescribed degree of observer stability can be ensured for all real α_{ij} by suitably choosing ε .

The form (C1) represents a generalization of an idea due to Moore and Ledwich [5] who employ a similar form for the diagonal blocks in the Bucy canonical form of D (rather than for D itself). Their canonical form, however, merely ensures stability.

It may be noted that the form (C1) excludes matrices D that are cyclic but have repeated eigenvalues with a multiplicity greater than two. However, this slight loss of generality is clearly of little practical significance. There is also an additional advantage in using the form (C1) in that the number of unknown parameters in D are reduced from p^2 to p.

Summarising, the stable observer problem becomes that of finding a pair (D,K) with D in the form (C1) such that (7-13) is satisfied.

7.3.2 Formulation as an optimization problem

Since it is unlikely that any arbitrary choice of (D,K) will satisfy (7-13), it is necessary to devise a computational procedure that will systematically update (D,K) until (7-13) is satisfied. This can be achieved by reformulating the stable observer problem as an optimization problem, as follows.

From (7-13), define a non-negative quadratic performance index

$$J(D,K) = \tfrac{1}{2}\,\mathrm{tr}\{[I_{n-r}-T_1'(T_1T_1')^{-1}T_1]F_1'F_1[I_{n-r}-T_1'(T_1T_1')^{-1}T_1]\}$$

which can be simplified to

$$J(D,K) = \tfrac{1}{2}\,\mathrm{tr}\{[I_{n-r}-T_1'(T_1T_1')^{-1}T_1]F_1'F_1\}\ . \qquad (7\text{-}15)$$

Notice that J(D,K) = 0 if and only if (7-13) is satisfied.

The observer parameters (D,K) may therefore be obtained by minimizing $J(D,K)$, with D restricted to be of the form (C1) in order to ensure a prescribed degree of stability. Once (D,K) are known, the remaining parameters (G,H,E) may be calculated from (7-4), (7-6) and (7-14).

The numerical minimization of $J(D,K)$ is, perhaps, best performed using a gradient algorithm such as the Davidon-Fletcher-Powell algorithm [9], or the algorithm of Marquardt [10] for sum-of-squares functions. This would require computable expressions for the various gradients of $J(D,K)$ and these are derived in the next section.

7.3.3 Evaluation of gradients

Notice that the minimization of J with respect to (D,K) is subject to the constraint (7-4). Therefore, form the Lagrangian $\mathcal{L}$, from (7-4) and (7-15), thus

$$\mathcal{L} = \tfrac{1}{2}\,\mathrm{tr}\{[I_{n-r}-T_1'(T_1T_1')^{-1}T_1]F_1'F_1\} + \mathrm{tr}\{(TA-DT-KC)L'\} \qquad (7\text{-}16)$$

where L is a $p\times n$ Lagrange multiplier matrix defined by

$$\frac{\partial \mathcal{L}}{\partial T} = LA'-D'L-(T_1T_1')^{-1}[T_1F_1'F_1(I_{n-r}-T_1'(T_1T_1')^{-1}T_1) \mid 0_{p,n-r}]$$

$$= 0_{p,n} \qquad (7\text{-}17)$$

Then, the expressions for the gradients of J with respect to D and K are

$$\frac{\partial J}{\partial D} = \frac{\partial \mathcal{L}}{\partial D} = -LT' \qquad (7\text{-}18)$$

$$\frac{\partial J}{\partial K} = \frac{\partial \mathcal{L}}{\partial K} = -LC' \qquad (7\text{-}19)$$

Since D is assumed to be in the special form (C1), the gradients of J with respect to the free parameters α_{ij} are required and these may be easily shown to be

$$\left.\begin{aligned}
&\left.\begin{aligned}
\frac{\partial J}{\partial \alpha_{i1}} &= 2\alpha_{i1}\varepsilon[LT']_{2i-1,2i} + 2\alpha_{i1}[LT']_{2i,2i} \\
\frac{\partial J}{\partial \alpha_{i2}} &= 2\alpha_{i2}[LT']_{2i-1,2i}
\end{aligned}\right\} i = \begin{cases} 1,\ldots,p/2, & p \text{ even} \\ 1,\ldots,(p-1)/2, & p \text{ odd.} \end{cases} \\
&\frac{\partial J}{\partial \alpha_{i}} = 2\alpha_{i1}[LT']_{p,p} \quad , \qquad\qquad i = (p+1)/2, \ p \text{ odd.}
\end{aligned}\right\} \tag{7-20}$$

where $[LT']_{k,j}$ denotes the k,j element of the matrix LT' .

Equations (7-4), (7-15), (7-17)-(7-20) provide sufficient information for the minimization of J using a gradient technique. This will be discussed further in §7.3.5.

7.3.4 Lower bound on stable observer order

It is, of course, unnecessary to start at p = 0 in designing the observer. In fact, the lower bound on p as contained in Lemma 1 of [3] may be used, i.e.,

$$p_1 = \text{rank } [F_1] \tag{B1}$$

A tighter bound than (B1) is also available (Lemma 3 of [3]). This involves the determination, usually by inspection, of the minimal rank of a Hankel matrix resulting from a realization procedure. The process is, however, tedious and requires the almost complete design of the observer itself.

An easily computable, though less tight, lower bound can be obtained through the following argument. There are $p^2 + p(r+l+n)+lr$ parameters in the 5-tuple (D,K,H,E,T), but, as pointed out in §7.3.1, these matrices are only unique to within an arbitrary $p \times p$ linear transformation. Thus p^2 of these parameters are redundant, and hence there are only $p(n+r+l)+lr$ independent parameters. Also, (7-4) and (7-5) constitute $n(p+1)$ equations to be satisfied. Therefore, a lower bound on the observer order is easily seen to be

$$p_2 \geqslant \frac{(n-r)l}{r+l} \tag{B2}$$

where it is understood that the right-hand side of (B2) is rounded up to an integer. Combining (B1) and (B2), the following new lower bound on the minimal order for a stable observer is obtained:

$$p'_{min} = \max[p_1 , p_2] \tag{7-21}$$

7.3.5 A design algorithm

The preliminaries having been completed, the following algorithm is now proposed for the design of a minimal-order observer having a prescribed degree of stability.

Step (i): Set $p = p'_{min}$ where p'_{min} is given in (7-21).

Step (ii): Choose an arbitrary pair (D,K) where D is in the form (C1) and the constant ε is chosen so as to obtain the desired degree of stability.

Step (iii): Compute $J, \left\{\frac{\partial J}{\partial \alpha_{ij}}\right\}, \frac{\partial J}{\partial K}$ using the results of §7.3.3.

Step (iv): Go to (v) if $\left\|\frac{\partial J}{\partial \alpha_{ij}}\right\|$ all i,j, $\left\|\frac{\partial J}{\partial K}\right\|$ are all less than some prescribed tolerances, thereby indicating that a local minimum of J has been reached. Otherwise update (D,K) in accordance with the rules for any gradient-type function minimization technique and return to step (iii).

Step (v): If the minimum value of J is less than some prescribed tolerance, then (D,K) constitute a partial solution to the stable observer problem, the complete solution being obtained by evaluating (G,H,E) from (7-4), (7-6) and (7-14). Otherwise either initialize a new search at greatly different starting values of (D,K) or increase the observer order p by one and return to step (ii).

Remarks

(R1). The reason for starting a new search at step (v) is that J, being a complex function of (D,K), may possess several local minima at which its value may be either finite or zero.

(R2). However, repeated failures to make J vanish would imply that an observer of the particular order being tried does not exist. It would then become necessary at step (v) to adopt the alternative course of increasing the observer order by one and trying again.

(R3). The proposed algorithm will definitely terminate before or when $p = n-r$, in which case the full-order Luenberger observer is obtained.

7.3.6 Examples

Two numerical examples are included to illustrate the design technique developed in the previous sections.

Example 1

A 5^{th}-order, 2-input, 2-output system taken from [3] is considered.

$$\overset{\circ}{x}(t) = \begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ -2 & 0 & 0 & 2 & -1 \\ 0 & 1 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 & -1 \\ -1 & 2 \\ -1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} u(t)$$

$$y(t) = \left[\begin{array}{c|c} 0_{2,3} & I_2 \end{array}\right] x(t)$$

It is required to estimate

$$Fx(t) = \tfrac{1}{2}\begin{bmatrix} -3 & -.5 & -.5 & -1 & -1 \\ 1 & -1.5 & .5 & -1 & -1 \end{bmatrix} x(t)$$

From (7-21), it is seen that the lower bound $p'_{min} = 2$. Hence the construction of a second-order observer is attempted.

First, in order to find an observer that is merely stable, a value of .001 is chosen for the decay parameter ε. Then following a procedure of §7.3.5 with a stopping criterion $J \leqslant 10^{-6}$, the solution shown in Table I is obtained. Since the observer has unsatisfactory poles, computation is repeated with $\varepsilon = 5$, and from Table I it is seen that the new solution does indeed have the desired degree of stability.

For this problem, the arguments in §7.3.4 show that there is an excess of two independent observer parameters over the number of conditions to be satisfied. Hence it may be expected that the solution for any ε is not unique and this is confirmed by further computational runs from different starting values of (D,K). To save space, the values of (G,H,E) are not shown in Table I, but can be easily obtained from equations (7-4), (7-6) and (7-14).

ε	Initial [D],[K]	No. of iterations	Final [D], [K]	Observer poles
.001	$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	12	$\begin{bmatrix} 0 & -.920 \\ 1 & -.002 \end{bmatrix}, \begin{bmatrix} 1.297 & 1.377 \\ 1.487 & .628 \end{bmatrix}$	$-.001 \pm j.960$
5	$\begin{bmatrix} 0 & -79 \\ 1 & -19 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	12	$\begin{bmatrix} 0 & -79.400 \\ 1 & -19.100 \end{bmatrix}, \begin{bmatrix} 1.453 & -.364 \\ 1.903 & .403 \end{bmatrix}$	-6.115,-12.985

Table I

Initial [D],[K]	No. of iterations	Final [D], [K]	Observer poles
$\begin{bmatrix} 0 & -25 \\ 1 & -16 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	13	$\begin{bmatrix} 0 & -1.000 \\ 1 & -.999 \end{bmatrix}, \begin{bmatrix} 2.821 & -.228 \\ 1.933 & 1.150 \end{bmatrix}$	$-.500 \pm j.866$
$\begin{bmatrix} 0 & -25 \\ 1 & -9 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	12	$\begin{bmatrix} 0 & -1.000 \\ 1 & -1.000 \end{bmatrix}, \begin{bmatrix} 3.007 & 1.008 \\ 3.995 & 1.994 \end{bmatrix}$	$-.500 \pm j.866$

Table II

Example 2

A second 5th-order, 2-input, 2-output plant is considered.

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & -3.5 \\ 0 & 0 & 0 & 1 & -1.5 \end{bmatrix} x(t) + \begin{bmatrix} -3 & 1 \\ 5 & 1 \\ -3 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} u(t)$$

$$y(t) = \left[0_{2,3} \;\vdots\; I_2 \right] x(t)$$

The aim is to design a second-order observer to estimate

$$Fx(t) = \begin{bmatrix} -1 & 1 & 0 & -1 & 2 \\ -1 & 0 & 1 & 0 & 1 \end{bmatrix} x(t)$$

Again, computations are first performed for $\varepsilon = .001$. It is found that although the observer obtained is not unique and depends on the starting values of (D,K), the observer poles are always the same. Two sample results are shown in Table II. Thus, unlike in Example 1, the freedom in positioning the observer poles is lost in this example. The essential difference between the two cases appears to be that the observability indices of the pair (A,B) are 2 and 3 for Example 1 and 4 and 1 for Example 2.

Clearly, for the present example there is no point in attempting solutions for other values of ε.

7.4 SOLUTION OF MINIMAL-ORDER FIXED-POLE OBSERVER PROBLEM

The solution to the fixed-pole observer problem can be similarly obtained using the procedure outlined in § 7.3.5, except that some minor modifications are required as follows:

In fixing all the observed poles, the matrix D in the form (C1) is completely specified. Consequently, the minimization procedure proposed in §7.3.5 is carried out only with respect to the parameters in K. However, the introduction of p-fixed poles is equivalent to a further reduction of p independent parameters as discussed in §7.3.4. Thus, a lower bound on p in this case is given by

$$p_3 \geq \frac{(n-r)l}{r+l-1} \tag{B3}$$

where the right-hand side of (B3) is again rounded up to an integer. Lastly, on combining (B1) and (B3), therefore the lower bound for the fixed-pole observer is

$$p''_{min} = \max\,[p_1\ ,\ p_3] \tag{7-22}$$

<u>Example 3</u>

Example 1 used in §7.3.6 is considered again. Suppose the observer is to have both poles at -4 (the pole configuration chosen by Roman and Bullock [3]), then (C1) yields

$$D = \begin{bmatrix} 0 & -16 \\ 1 & -8 \end{bmatrix}$$

Subsequent minimization of J with K as the only unknown parameter matrix also results in non-unique solutions, e.g., the two cases shown in Table III.

Initial K	No. of iterations	Final K
$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	12	$\begin{bmatrix} -3.901 & -4.793 \\ 12.025 & 2.802 \end{bmatrix}$
$\begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix}$	10	$\begin{bmatrix} 3.285 & -1.496 \\ 9.825 & 2.386 \end{bmatrix}$

Table III

7.5 THE OBSERVER AS A DYNAMIC COMPENSATOR

When the reduced-order observer output w(t) is fed back directly to the plant input u(t) (see Fig. 7.4), the observer may now be regarded as a dynamic compensator.

In this section, the suitability of such a control scheme will be examined in cases where w(t) (asymptotically) approaches the linear state feedback law Fx(t) resulting from one of the following design objectives:

(i) minimization of the usual quadratic performance index, or

(ii) exact pole placement, or

(iii) decoupling of the LMS (7-1).

Consequently, the control scheme shown in Fig. 7.4 can be regarded as a possible alternative to the more direct output feedback controller design techniques described earlier in Chapters 2 and 4.

Notice that in setting u(t) = w(t) in (7-1) and (7-2), the resulting closed-loop composite system can be described by the differential equation

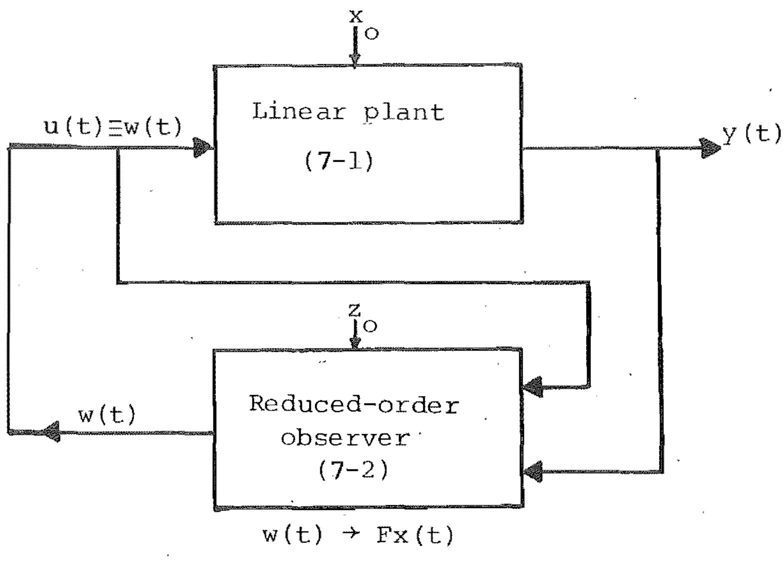

Fig. 7.4 Observer as dynamic compensator

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A + BEC & BH \\ KC + GEC & D + GH \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} , \quad \begin{bmatrix} x(0) \\ z(0) \end{bmatrix} = \begin{bmatrix} x_o \\ z_o \end{bmatrix} \qquad (7-22)$$

It can also be readily shown [2] that the closed-loop poles of the system (7-22) are actually those contained in $A+BF$ and D.

7.5.1 Implementation of optimal state feedback law

The optimal linear control law for plant (7-1) that minimizes the quadratic performance index

$$J = \tfrac{1}{2} \int_0^\infty x'(t)Qx(t) + u'(t)Ru(t)dt \qquad (7-23)$$

with $Q \geqslant 0$, $R > 0$ is well known [13] and is given by

$$u(t) = u_{opt}(t) = F_{opt}\, x_{opt}(t) \qquad (7-24)$$

where

$$F_{opt} = -R^{-1}B'P \qquad (7-25)$$

$P = P' > 0$ is the solution of the algebraic Riccati equation

$$PBR^{-1}BP - PA - A'P - Q = 0 \qquad (7-26)$$

The optimal solution is unique. Substitution of (7-25) into (7-1) results in the optimal state trajectory

$$x_{opt}(t) = e^{(A+BF_{opt})t}\, x(0) \qquad (7-27)$$

and the minimum cost is given by

$$J_{min} = \tfrac{1}{2}\, x_o'\, Px_o \qquad (7-28)$$

The performance of the closed-loop system (7-22) will now be assessed in terms of the same quadratic cost function

J given by (7-23). First, define an error signal e(t) where $z(t) \triangleq Tx(t) + e(t)$. Then, from (7-4) and (7-5), it can be easily verified that

$$\left.\begin{aligned} \dot{e}(t) &= De(t) \\ \text{with} \qquad e(0) &= z_o - Tx_o \end{aligned}\right\} \qquad (7\text{-}29)$$

Since w(t) only approaches $u_{opt}(t)$ asymptotically, the resulting state trajectory x(t) will be different from $x_{opt}(t)$. Thus define

$$e_x(t) \triangleq x(t) - x_{opt}(t) \qquad (7\text{-}30)$$

It can be readily seen that

$$\dot{e}_x(t) = (A + BF_{opt})e_x(t) + BHe(t) \qquad (7\text{-}31)$$

Now, following an analysis similar to that described in [11] and using (7-26) and (7-31), the difference between J and J_{min} is found to be

$$\delta J \triangleq J - J_{min} = \tfrac{1}{2}\int_0^\infty e'(t)H'He(t)dt \qquad (7\text{-}32)$$

In general, $He(t) \neq 0$ unless at least one of the following conditions is satisfied:

(a) $H \equiv 0$

(b) $e(t) \equiv 0$

However, condition (a) is unlikely to occur because this implies that w(t) does not contain z(t). Also, condition (b) is satisfied if and only if $e(0) \equiv 0$. This requires x_o to be known exactly which is seldom true in practice.

Hence in using the present control scheme, $\delta J > 0$. A similar result has been demonstrated in [11] for the special case $p = n-r$ (i.e., the Luenberger observer).

Note that δJ depends on $e(t)$ which in turn is determined by the following factors:

(i) the decay rate of $e(t)$, i.e. the degree of stability of $\lambda_i(D)$, and

(ii) the magnitude of e_o, i.e. the mismatch between z_o and Tx_o.

In view of (ii), this means that δJ cannot necessarily be made arbitrarily small by only making $Re\{\lambda_i(D)\}$ sufficiently negative. This point will be demonstrated by a numerical example later.

Finally, in order to devise a simple computational expression for δJ, suppose x_o is treated as a zero mean random variable with known covariance (see Chapter 2), i.e.

$$E\{x_o\} = 0 \quad , \qquad E\{x_o x_o'\} \triangleq X_o$$

Then, from theorem 1, §2.2.3, (7-32) becomes

$$\delta J = \tfrac{1}{2}\, tr\{\Pi E_o\} \qquad (7\text{-}33)$$

where $\Pi = \Pi'$ is the positive definite solution of

$$D'\Pi + \Pi D + H'H = 0 \qquad (7\text{-}34)$$

and $E_o = E\{e(0)e'(0)\}$. For practical reasons (see also §2.2.1), it may be assumed that $z_o = 0$. In which case, $E_o = TX_oT'$.

The results of this section will be now be applied to a numerical example.

Example

Consider the plant

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [1 \quad 0]\; x(t)$$

Minimization of the quadratic performance index

$$J = \tfrac{1}{2} \int_0^\infty x'(t) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x(t) + u^2(t)dt$$

results in the optimal state feedback gain matrix

$$F_{opt} = [-.4 \quad -.9]$$

Suppose an observer is to be constructed to estimate the optimal state feedback law. From the lower bounds given in earlier sections, it is seen that the observer must be of order 1 which is actually a Luenberger observer. Therefore, the observer pole $\lambda(D)$ may be fixed at any arbitrary value. Two sets of results corresponding to two different values of $\lambda(D)$ are shown in Table IV.

$\lambda(D)$	(D, K, H, E)	δJ
-10	(-10, 10.07, -4.01, 3.10)	0.32
-100	(-100, 55.13, -72.56, 39.10)	3.88

Table IV

Thus it is obvious that making $Re\{\lambda(D)\}$ sufficiently negative does not necessarily result in a corresponding reduction in δJ.

For comparison purposes, a first-order dynamic compensator has also been constructed via the design technique described in Chapter 2. The optimal compensator obtained is

$$\dot{z}(t) = -1.414\ z(t) + 33.29\ y(t)$$

$$u(t) = -0.081\ z(t) - 0.496\ y(t)$$

and

$$\delta J < 10^{-2}.$$

Thus the performance of the optimal compensator is far more satisfactory than that of the observer-compensator, which is however of the same order.

Notice that in general

(i) the order of the observer-compensator is bounded below by equation (7-21) or (7-22), whereas the order of the dynamic compensator considered in Chapter 2 can assume any value p, $0 \leqslant p \leqslant n-r$, provided the resulting closed-loop system is stable,

(ii) with feedback via an observer, n closed-loop poles always coincide with the eigenvalues of $A + BF$. This structural rigidity may partly account for the relatively poor performance of observer-compensator.

7.5.2 Pole placement

For systems with inaccessible states, it has been proposed (see e.g. [2]) that the observer be used to implement the state feedback law obtained for pole placement. However, it is well known [15] that the state feedback law is, in general, non-unique. Hence, the order of one observer-compensator may be higher than that of another although in both cases the same pole configuration $\lambda_i(A+BF)$, $i = 1,\ldots,n$ is obtained.

Also, the order of the observer may be unnecessarily high; for instance, consider the 4^{th} order, 2-input, 2-output example of §4.2.5. For almost all pole configuration, the lower bounds of earlier sections indicate that the observer-compensator must be of order 2. However, the more direct pole placement technique of Chapter 4 requires only constant output feedback.

7.5.3 Decoupling

If the LMS (7-1) is decoupled using the control law

$$u(t) = Fx(t) + Lv(t) \tag{7-35}$$

where v(t) is an external new input, it has also been proposed that (see e.g. [16]) if some of the state variables are not directly measurable, an observer may be used to implement the state feedback part of (7-35). Unfortunately, the decoupling law (7-35) is also non-unique in general (see e.g. [17]) and the same criticism made earlier in §7.5.2 concerning the observer order also applies in this case.

7.6 CONCLUSION

The design of minimal-order observers for asymptotically estimating vector functions of the states of linear multivariable systems was first considered. By utilizing the concept of projection matrix [8], a key necessary condition governing an observer's existence was simplified. The design problem was then transformed into a static optimization problem in certain observer parameters; and the required expressions were derived for obtaining a solution using gradient techniques. A systematic procedure was then proposed which, in conjunction with a new lower bound on the observer order developed in this chapter, may be employed for the design of minimal-order stable observers. Observer stability is ensured by adopting the special canonical form (C1) of §7.3.1; in fact any prescribed degree of asymptotic stability may be guaranteed by the choice of a single parameter. The same procedure may also be used, with only slight modifications, for the design of minimal-order fixed-pole observers.

Although some insight is lost when using the proposed technique, it has an important advantage over other existing techniques in that it is conceptually simple and can be easily programmed on a digital computer. Moreover, the procedure has the added attraction that the observer structural indices, so vital in the approaches of [3] and [5], need not be considered.

Consideration was also given to the use of an observer to implement the state feedback law that is optimal with respect to a quadratic performance index. It was shown that this always results in an increase in cost which cannot necessarily be made arbitrarily small merely by making the real parts of the observer poles sufficiently negative. Numerical results were also presented to show that the observer-compensator may even be considerably inferior in performance to the optimal dynamic compensator of Chapter 2.

Finally, the possibility of using the same control scheme for the purpose of pole placement or decoupling has also been discussed. On the basis of the compensator order required to achieve the design objective, it is concluded that again the more direct techniques described in Chapter 4 and [17] are to be preferred.

REFERENCES

[1] Luenberger, D.G., "Observers for multivariable systems," IEEE Trans. Automat. Contr., vol. AC-11, pp.190-197, Apr. 1966.

[2] Fortmann, T.E. and Williamson, D., "Design of low-order observers for linear feedback control laws," IEEE Trans. Automat. Contr., vol. AC-17, pp.301-308, June 1972.

[3] Roman, J.R. and Bullock, T.E., "Design of minimal order stable observers for linear functions of the state via realization theory," IEEE Trans. Automat. Contr., vol. AC-20, pp.613-622, Oct. 1975.

[4] Anderson, B.D.O., Bose, N.K. and Jury, E.I., "Output feedback stabilization and related problems - solution via decision methods," IEEE Trans. Automat. Contr., vol. AC-20, pp.53-65, Feb. 1975.

[5] Moore, J.B. and Ledwich, G.F., "Minimal order observers for estimating linear functions of a state vector," IEEE Trans. Automat. Contr., vol. AC-20, pp.623-632, Oct. 1975.

[6] Molinari, B.P., "Algebraic solution of matrix linear equations in control theory," Proc. I.E.E., vol. 116, pp.1748-1754, Oct. 1969.

[7] Gantmacher, F.R., The Theory of Matrices, I. New York: Chelsea, 1959.

[8] Rosen, J.B., "The gradient projection method for non-linear programming," SIAM J. Appl. Maths, vol. 8, 181, 1960.

[9] Fletcher, R. and Powell, M.J.D., "A rapidly convergent descent method for minimization," British Computer J., vol. 6, pp.163-168, 1963.

[10] Marquardt, D.W., "An algorithm for least squares estimation of non-linear parameters," SIAM J., vol. 11, pp.431-441, 1963.

[11] Bongiorno, J.J., Jr. and Youla, D.C., "On observers in multivariable control systems," Int. J. Contr., vol. 8, pp.221-243, Sept. 1968.

[12] Luenberger, D.G., "Observing the state of a linear system," IEEE Trans. Mil. Electron., vol. MIL-8, pp.74-80, Apr. 1964.

[13] Anderson, B.D.O. and Moore, J.B., Linear Optimal Control. Englewood Cliffs, N.J.: Prentice-Hall, 1971.

[14] Sirisena, H.R. and Choi, S.S., "An algorithm for constructing minimal-order observers for linear functions of the state," Int. J. System Science, to appear.

[15] Wonham, H.W., "On pole assignment in multi-input controllable linear systems," IEEE Trans. Automat. Contr., vol. AC-12, pp.660-665, Dec. 1967.

[16] Majumdar, A.K. and Choudhury, A.K., "On the decoupling of linear multivariable systems," Int. J. Contr., vol. 17, pp.225-256, Feb. 1973.

[17] Sirisena, H.R. and Choi, S.S., "Minimal-order compensators for decoupling and arbitrary pole placement in linear multivariable systems," Int. J. Contr., to appear.

CHAPTER 8

CONCLUSION

8.1 CONTRIBUTIONS MADE IN THIS THESIS

The main objective of the research project has been to develop efficient design techniques for the control of LMS using only the directly measurable system variables. Essentially, three distinctively different design philosophies have been examined, namely

(a) the optimal control approach wherein a quadratic performance index in the state and control variables is minimized,

(b) the pole placement approach which involves either exact or approximate placement of the system closed-loop poles at suitable positions in the complex plane, and

(c) the observer-compensator approach where the observer is constructed to realise a suitable state feedback law.

Contributions made in this thesis to each of the above areas are now recapitulated.

8.1.1 Optimal output feedback approach

Although a state-space approach to the design of optimal fixed-configuration dynamic compensators for essentially noise-free LMS has been reported in several publications, the indirect-type solution techniques proposed there are either computationally expensive or numerically unstable. In contrast, the direct gradient-type solution technique proposed in Chapter 2 is more efficient computationally because (a) it avoids the solution of non-linear equations, and (b) it appears to exhibit rapid convergence. Consequently, the design of optimal, fixed-configuration dynamic compensators via a state-space formulation has become a more viable proposition. For cases where the LMS is open-loop unstable, a modified gradient-type solution technique has been developed, also in Chapter 2, whereby the difficulty of choosing a stabilizing compensator to initialize the design procedure has been avoided.

In Chapter 3, the approach of Chapter 2 has been extended to LMS with appreciable amounts of process and observation noise. This represents a unification and generalization of previous work in this area. Both stationary and non-stationary stochastic problems have been considered. In either case, necessary expressions have been derived and a direct gradient-type solution technique, not unlike that described in Chapter 2, has been proposed. Apart from its computational efficiency, the gradient-type solution technique has an added advantage over existing indirect-type solution techniques because in the case of the non-stationary stochastic control problem, whereas the indirect-type techniques fail due to the singularity of the variational problem, the direct-type solution technique, being based on a standard conjugate gradient algorithm, in principle has no difficulty in obtaining the solution.

For ease of implementation, a computational technique has also been devised for the design of suboptimal compensators whose gains are constrained to be a piecewise-constant function of time, with provision for optimally choosing the instants at which the gains change.

8.1.2 Pole placement approach

A new unrestricted-rank pole placement technique has been developed in Chapter 4. Unlike in the work of most other researchers, no restrictions have been imposed on the ranks of the compensator matrices. The result of this increased design flexibility also means that exact pole placement is always achieved with a compensator of lowest possible order. A computationally efficient method has also been developed, in Chapter 4, for the design of output feedback control systems with the closed-loop poles constrained to lie in prescribed regions of the complex plane. The compensators obtained are usually of lower order than that required for exact pole placement.

Consideration has also been given to the more general problem of achieving either exact or approximate pole placement while minimizing a second performance index having one of the following forms:-

(i) A quadratic performance index in the state and control variables. This problem formulation, described in Chapter 4, permits the simultaneous attainment of several design objectives such as obtaining pole placement while either optimizing the system response or minimizing the mean square control effort.

(ii) A measure of the sensitivity of the closed-loop poles to plant parameter changes. This problem has been considered in Chapter 5 where two design approaches have been developed. In the first approach, exact pole placement is obtained at the nominal values of the plant parameters while minimizing a measure of the sensitivity of the poles to small parameter variations. In the second approach, the plant parameters are permitted to vary over a range that is large compared to their nominal values and the design obtained gives approximate pole placement in a mean square error sense over the prescribed range of plant parameter variations.

(iii) A measure of the steady-state error due to the presence of constant measurable and unmeasurable disturbance inputs. This servomechanism design problem has been considered in Chapter 6. For unmeasurable constant disturbances, the proposed technique results in the closed-loop systems having both minimum mean-square steady-state error and a desired transient response characteristic through suitable positioning of the closed-loop poles. A new technique has also been developed for the design of the minimal-order compensator for achieving both zero steady-state error and exact pole placement. Unlike previous work in this area, the use of complete state feedback laws is avoided. Also, provided the two conditions stated in Chapter 6 are satisfied, it has been shown how the steady-state error can be nulled using a feedforward controller. Finally, the design of linear multivariable servomechanisms with randomly time-varying disturbances has also been considered. A quadratic performance index has been adopted for which a solution technique, similar to that described in Chapter 3 for the stationary stochastic control problem, has been proposed. Satisfactory performance of such servomechanisms can thereby be attained without the use of optimal controllers incorporating high-order state estimators.

8.1.3 Observer-compensator approach

In Chapter 7, attention has been focussed firstly on the design of minimal-order stable or fixed-pole observers for asymptotic estimation of vector functions of the state of LMS. By geometric arguments, the design problem has been formulated into a parameter optimization problem in certain observer parameters, and the required expressions have been derived for obtaining a solution using gradient techniques. Next, a systematic procedure has been proposed which, in conjunction with the new lower bounds on the observer order developed, can be employed for the design of minimal-order stable or fixed-pole observers. The suitability of using the observer to implement state feedback laws resulting either from optimal control, pole placement or decoupling has also been examined. For reasons stated there, it is concluded that such a control scheme is inferior when compared to the more direct approaches described in Chapters 2, 4 and [1].

8.2 POSSIBLE AREAS FOR FURTHER RESEARCH

Although the various design techniques developed in this thesis have been tested on numerical examples, the potential of each of these techniques, or combinations of them, can only be assessed through application to practical design problems. Most likely, it will be necessary to tailor the techniques to suit the particular application. Thus, this should logically be the next major stage of the research program.

The following points that have arisen in the course of the present investigation may also be the subject of further research.

It will be interesting to prove (or disprove) the conjecture made in §3.3.3 concerning the absence of impulsive control although, as has been pointed out, the outcome of this investigation will not affect the applicability of the method proposed there in so far as the implementation of the control law is concerned.

Despite recent efforts [2], [3], no explicit expression has yet been reported on the minimum compensator orders required for (a) the stabilization of a given open-loop unstable system and (b) exact placement of all the closed-loop system poles, and this remains a fruitful area for research.

Extension of the techniques described in §6.2 to include more general types of disturbance inputs such as polynomials or exponentials is another possibility.

Finally, it may also be worth looking into the refinement of the observer design algorithm of Chapter 7 by including consideration of structural indices [4].

REFERENCES

[1] Sirisena, H.R. and Choi, S.S., "Minimal-order compensators for decoupling and arbitrary pole placement in linear multivariable systems," Int. J. Contr., accepted for publication.

[2] Kimura, H., "Pole assignment by gain output feedback," IEEE Trans. Automat. Contr., vol. AC-20, pp.509-516, Aug. 1975.

[3] Anderson, B.D.O., Bose, N.I. and Jury, E.I., "Output feedback stabilization and related problems - solution via decision methods," IEEE Trans. Automat. Contr., vol. AC-20, pp.53-66, Feb. 1975.

[4] Moore, J.B. and Ledwich, G.F., "Minimal order observers for estimating linear functions of a state vector," IEEE Trans. Automat. Contr., vol. AC-20, pp.623-632, Oct. 1975.